Group-Variation Equations for the Coefficients in the $1/N$ Expansions of Physical Quantities in SU(N) Gauge Theories in $D = 3 + 1$

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Abstract

The coefficients in the $1/N$ expansions of the vacuum expectation values and correlation functions of Wilson loops, in continuum SU(N) gauge theories in 3+1 dimensions, are shown to be determined by a closed and complete set of equations, called the Group-Variation Equations, that exhibit a simple and robust mechanism for the emergence of massive glueballs and the Wilson area law. The equations predict that the cylinder-topology minimal-area spanning surface term in the two-glueball correlation function, when it exists, must be multiplied by a pre-exponential factor, which for large area A of the minimal-area cylinder-topology surface, decreases with increasing A at least as fast as $\frac{1}{\ln(\sigma A)}$. If this factor decreases faster than $\frac{1}{\ln(\sigma A)}$, then the mass $m_{0^{++}}$ of the lightest glueball, and the coefficient σ of the area in the Wilson area law, are determined in a precisely parallel manner, and the equations give a zeroth-order estimate of $m_{0^{++}}/\sqrt{\sigma}$ of 2.38, about 33% less than the best lattice value, without the need for a full calculation of any of the terms in the right-hand sides. The large distance behaviour of the vacuum expectation values and correlation functions is completely determined by terms called island diagrams, the dominant contributions to which come from islands of *fixed* size of about $\frac{1}{\sqrt{\sigma}}$. The value of σ is determined by the point at which $\left|\frac{\beta(g)}{g}\right|$ reaches a critical value, and since the large distance behaviour of all physical quantities is determined by islands of the *fixed* size $\frac{1}{\sqrt{\sigma}}$, the running coupling g^2 never increases beyond the value at which $\left|\frac{\beta(g)}{g}\right|$ reaches the critical value. Evidence is given, based on 't Hooft's demonstrations of the geometric convergence of sums of planar diagrams, and the fact that the coefficients in $\beta(g)$ all have the same sign, in natural renormalization schemes, that the sums in the right-hand sides of the Group-Variation Equations will converge geometrically, and that the critical value of g^2 will be *strictly smaller* than the radius of convergence, which would imply that the Group-Variation Equations provide a basis for the calculation of all physical quantities within the framework of the $1/N$ expansion, if a systematic method exists for solving the equations, for example, the iterative substitution of the left-hand sides into the right-hand sides.

Contents

Introduction

The continuing desirability of a reliable analytic method of calculating physical quantities, at large distances, in QCD, has recently been emphasized by Gross, [1], [2], and Witten, [3]. In this paper I give a new, complete, set of equations, called the Group-Variation Equations, for the coefficients in the $\frac{1}{N}$ expansions of vacuum expectation values and correlation functions of Wilson loops in SU(N) Yang-Mills theory in $3+1$ dimensions, which exhibit a simple and robust mechanism for the emergence of massive glueballs and the Wilson area law.[1] There is a separate Group-Variation Equation for each non-vanishing coefficient in the $\frac{1}{N}$ expansion of a vacuum expectation value or correlation function, and the Group-Variation Equation for a coefficient expresses the derivative of that coefficient, with respect to the coupling constant g^2, in a natural way, in terms of that coefficient itself, and the other coefficients, of the same, or lower, order in the $\frac{1}{N}$ expansions.

If we consider a set of one or more Wilson loops, and define a one-parameter family of sets of loops, by multiplying the sizes and separations of the loops in the original set by a scale factor L, then by the renormalization group, the Group-Variation Equation for the vacuum expectation value, or the correlation function, of any set of loops in the family, can be expressed as an equation for the derivative with respect to L, of that vacuum expectation value or correlation function. The Group-Variation Equation, for that vacuum expectation value or correlation function, can then be integrated with respect to L, starting from boundary conditions at small L, as given by renormalization-group-improved perturbation theory, and continuing to *arbitrarily large* L, since the structure of the Group-Variation Equations guarantees that the Wilson area law, and the massive glueball saturation of the correlation functions, solve the equations self-consistently, at long distances.

The Group-Variation Equations for the lowest non-vanishing coefficients in the $\frac{1}{N}$ expansions of the vacuum expectation value of one Wilson loop, and the correlation functions of two or more Wilson loops, close among themselves, and their right-hand sides have a simple representation in terms of planar diagrams, where a simply connected window in a diagram represents the vacuum expectation value of the Wilson loop, defined on the closed loop of paths that forms the border of the window, a multi-

[1] By a Wilson loop, I mean the trace of a general closed-loop path-ordered phase factor, not just rectangular loops. We perform a standard Wick rotation to 4 Euclidean dimensions.

ply connected window represents the correlation function of the Wilson loops, defined on the closed loops of paths that form the border of the window, and the propagators in the diagrams represent sums over paths, that are to be taken with kinematic weights given explicitly in the paper. [2]

Each diagram is multiplied by an integer coefficient, $\frac{\mathrm{d}}{\mathrm{d}M}\mathbf{C}(M)\Big|_{M=1}$, where $\mathbf{C}(M)$ is the chromatic polynomial of the diagram, which is the number of distinct ways of colouring the windows of the diagram with M colours available, subject to the map-colouring rule that no two windows that meet along a propagator, are coloured the same colour. This coefficient *vanishes* for the vast majority of diagrams, because their chromatic polynomial, $\mathbf{C}(M)$, contains two or more factors of $(M-1)$.

The long-distance behaviour of the vacuum expectation value of one Wilson loop, and the correlation functions of two or more Wilson loops, is completely controlled by the diagrams that have an island, or in other words, a connected group of propagators, none of which joins onto any of the Wilson loops in the left-hand side of the equation. The requirement that $\frac{\mathrm{d}}{\mathrm{d}M}\mathbf{C}(M)\Big|_{M=1}$ be non-vanishing, implies that a diagram can have at most one island, and, if it has an island, it can have *no* propagator that does not form part of the island.

In particular, an island diagram, in the Group-Variation Equation for the vacuum expectation value of one Wilson loop, has one window of cylinder topology, that represents the correlation function of the left-hand side Wilson loop, and the Wilson loop defined on the closed loop of paths that forms the outer border of the island, and one or more simply connected windows inside the island, that represent the vacuum expectation values of the Wilson loops defined on the closed loops of paths that form their borders. This structure results in the emergence of the Wilson area law in a simple and robust way: the window weights suppress long propagators, essentially by generating an effective mass for the propagators in their borders. Thus, once some semblance of an area law has appeared, the window weights, for the internal windows of the island, suppress the contributions of all islands of size larger than $\frac{1}{\sqrt{\sigma}}$, where σ is the coefficient of the area, in the emerging area law. An island diagram will then give a contribution proportional to the area of the minimal-area spanning surface of the left-hand side Wilson loop, because we may expect a comparable contribution to

[2] Window-weighted path integrals were studied by Migdal and Makeenko [37], in connection with the analysis of the Migdal-Makeenko equations, but the Group-Variation Equations have no connection with the Migdal-Makeenko equations.

the island diagram, from an island of size about $\frac{1}{\sqrt{\sigma}}$, that is situated close enough to any point of the minimal-area spanning surface of the left-hand side Wilson loop.

Applying $L\frac{\partial}{\partial L}$ to the left-hand side Wilson loop also results in a factor of the area of the minimal-area spanning surface of the loop, thus verifying the consistency of the area law. This is called the "island diagram mechanism". A similar argument shows that the island diagrams are also responsible for the massive glueball saturation of the correlation functions, provided the mass of the lightest glueball is less than twice the effective mass generated for the propagators by the window weights.

The group-variation equations are derived by expressing the vacuum expectation values and correlation functions, for SU(NM), first in terms of those for $(\mathrm{SU}(N))^M$, then in terms of those for SU(N), differentiating with respect to M and setting $M = 1$, and in the left-hand sides, expressing the derivative with respect to M, in terms of a derivative with respect to g^2, then in terms of a derivative with respect to the sizes and separations of the Wilson loops involved, via the renormalization group.

To motivate the Group-Variation Equations, crudely, as a way of summing the planar diagrams of large-N Yang-Mills theory, consider the sum of all the planar Feynman diagrams that contribute, at leading non-vanishing order in $\frac{1}{N}$, to the vacuum expectation value of one Wilson loop, or to the correlation function of two or more Wilson loops. They are the planar diagrams that can be drawn on a sphere with n holes, or a disk with $n-1$ holes, where n is the number of Wilson loops involved. Imagine colouring each window, of each diagram, independently, with any of M possible colours. This results in taking each diagram a total of M^w times, where w is the number of windows of the diagram, which means that, up to an overall power of M, that is the same for all the diagrams, we are now calculating the vacuum expectation value, or correlation function, for SU(NM), with coupling constant g, or alternatively, for SU(N), with coupling constant $g\sqrt{M}$, rather than for SU(N), with coupling constant g. Now draw thick lines, or "borders", along all propagators that run between two windows coloured in two *different* colours, and group together all coloured diagrams that have the same pattern of coloured "countries", if we ignore the internal lines inside the coloured "countries". We then see that, if we look at the internal lines inside the coloured "countries", in such a group of coloured diagrams, and ignore distinctions between gluon lines and Fadeev-Popov lines, (which are dealt with in detail in the paper), we have, for each simply-connected coloured "country", all the planar Feynman diagrams that contribute, in leading non-vanishing order in the $\frac{1}{N}$ expansion, to the

vacuum expectation value of the Wilson loop defined on the "path" around the border of the "country", and for each non-simply-connected coloured "country", all the planar Feynman diagrams that contribute, in leading non-vanishing order in the $\frac{1}{N}$ expansion, to the correlation function of the Wilson loops defined on the "paths" around the two, or more, connected components of the border of that "country". Here we have focussed on specific paths, in the sums over paths that build up the propagators around the borders of the "countries", and we then have to sum over these paths, weighted by the appropriate kinetic factors, which are given in the paper. The window weights for these "countries" are in SU(N) Yang-Mills theory, with coupling constant g.

Now add together all the sets of coloured Feynman diagrams that correspond to the same pattern of "countries", but coloured in different ways, with M different colours available, subject to the map-colouring rule that no two "countries", that share a common border, are coloured in the same colour. This simply results in multiplying the contribution of any one of the possible colourings, by $\mathbf{C}(M)$, where $\mathbf{C}(M)$, the chromatic polynomial of the pattern of "countries", is the number of different possible colourings of the pattern of countries, with M different colours available, subject to the map-colouring rule. We can now differentiate with respect to M and set M equal to 1, which gives the factor $\frac{\mathrm{d}}{\mathrm{d}M}\mathbf{C}(M)\Big|_{M=1}$. Each pattern of "countries", for which this factor is non-vanishing, gives a diagram that contributes to the right-hand side, of the group-variation equation, for the vacuum expectation value, or correlation function, that we started with, provided we again ignore distinctions between gluon and Fadeev-Popov lines, which are treated in detail in the paper.

It is not hard to see, in a crude way, that if, in SU(NM) Yang-Mills theory, we integrate over the gauge fields *outside* the $(\mathrm{SU}(N))^M$ subgroup, for fixed values of the fields *in* the $(\mathrm{SU}(N))^M$ subgroup, which are treated as background fields, then for large N, at fixed M, we get planar diagrams, each window of which is associated with one of the M different SU(N) subgroups of $(\mathrm{SU}(N))^M$, such that any two windows that share a common border, (i.e. meet along a propagator), must be in *different* SU(N) subgroups of $(\mathrm{SU}(N))^M$. This is because, if we view the SU(NM) gauge fields as entries in an $NM \times NM$ matrix, each gauge field, outside the block-diagonal $(\mathrm{SU}(N))^M$ subgroup, interacts with precisely two *different* SU(N) subgroups, of the block-diagonal $(\mathrm{SU}(N))^M$ subgroup. In the planar diagram picture, the two distinct SU(N) subgroups of $(\mathrm{SU}(N))^M$, that a non-block-diagonal gauge field interacts with, can be viewed as associated with the two "sides" of the propagator, of that non-block-diagonal gauge

field.

The windows of these planar diagrams do not have to be simply-connected. In particular, a window can look like a "lake", with one or more "islands" in it. The only requirement is that the windows, on the two sides of a propagator, must be in *different* $\mathrm{SU}(N)$ subgroups of $(\mathrm{SU}(N))^M$, and this implies, in particular, that no "island" can be connected to the "shore" of a "lake" by a single "causeway", (i.e. a single propagator), because both sides of that propagator, being two different parts of the edge of a single window, would then be in the *same* $\mathrm{SU}(N)$ subgroup of $(\mathrm{SU}(N))^M$.

If we now use the general realization of propagators in background gauge fields, as sums over paths, weighted by path-ordered phase factors, calculated in those background gauge fields, we then see that, still considering the block-diagonal $(\mathrm{SU}(N))^M$ gauge fields as fixed background fields, we get planar diagrams, in which the propagators represent sums over paths, each weighted by the product of two path-ordered phase factors, one in each of the two different $\mathrm{SU}(N)$ subgroups of $(\mathrm{SU}(N))^M$, associated with the windows, on the two sides of that propagator, in the diagram. The path-ordered phase factors, around the border of any window, are all in the same $\mathrm{SU}(N)$ subgroup of $(\mathrm{SU}(N))^M$, and the traces of their products, around the connected components of the border of that window, are gauge-invariant.

We now, in the limit of large N, at fixed M, integrate over the block-diagonal $(\mathrm{SU}(N))^M$ gauge fields. Then in view of the factorization of vacuum expectation values of products of Wilson loops at leading order in large N, we will find that for each simply-connected window, and fixed paths in the sums over paths in the propagators at the edges of that window, we simply get the vacuum expectation value, in $\mathrm{SU}(N)$, of the Wilson loop formed by the paths round the edge of that window. This is of course completely independent of *which* of the M $\mathrm{SU}(N)$ subgroups of $(\mathrm{SU}(N))^M$ is associated with that window.

It is not so obvious that for a non-simply-connected window, firstly, all the two or more Wilson loops around its perimeter must be in the *same* $\mathrm{SU}(N)$ subgroup of $(\mathrm{SU}(N))^M$, and secondly, we get the leading non-vanishing, (at large N), contribution to the *correlation function*, in $\mathrm{SU}(N)$, of the Wilson loops that form its perimeter. That this is so, is shown in detail in the first half of this paper. A crude way of seeing that the Wilson loops around the perimeter of a non-simply connected window must be in the same $\mathrm{SU}(N)$ subgroup of $(\mathrm{SU}(N))^M$, is to consider the planar *Feynman* diagrams that can contribute to that window, as discussed above, and note that the

distinct SU(N) subgroups of $(\mathrm{SU}(N))^M$ do not interact with one another.

Finally, we sum over all distinct assignments of the different SU(N) subgroups of $(\mathrm{SU}(N))^M$ to the windows of our diagram, subject to the "map-colouring" rule that no two adjacent windows share the same one of the M different "colours". The total number of distinct assignments of M different "colours" to the windows of the diagram, subject to the map-colouring rule, is given by the "chromatic polynomial" of the diagram, $\mathbf{C}(M)$, which is easily calculated.

We thus find that, at leading non-vanishing order at large N, the vacuum expectation values and correlation functions of Wilson loops, in SU(NM) Yang-Mills theory, are expressed in terms of the same vacuum expectation values and correlation functions, in SU(N) Yang Mills theory. But at leading non-vanishing order at large N, these same vacuum expectation values and correlation functions, in SU(NM) Yang Mills theory, are also related to those of SU(N) Yang Mills theory, by a rescaling of the SU(N) coupling constant: $g \to g\sqrt{M}$. Thus we have expressed the leading non-vanishing contributions, at large N, to the vacuum expectation values and correlation functions of Wilson loops, in SU(N) Yang Mills theory, with the coupling constant $g\sqrt{M}$, in terms of those same vacuum expectation values and correlation functions, in SU(N) Yang Mills theory, with the coupling constant g, in such a way that the only dependence of the right-hand sides of the equations on M, is through the chromatic polynomials $\mathbf{C}(M)$.

We can thus differentiate with respect to M, then set M equal to 1. The effect on the left-hand sides of the equations is that we now have the derivative, with respect to g, of the vacuum expectation values and correlation functions, which may be further expressed, via the renormalization group, in terms of the derivative of those vacuum expectation values and correlation functions, with respect to an overall length-scale factor, if the sizes and separations of all the Wilson loops involved, are re-scaled by a common scaling factor.

The effect on the right-hand sides of the equations is to produce a vast reduction in the number of diagrams involved, because for huge classes of diagrams, the chromatic polynomial $\mathbf{C}(M)$ has two or more factors of $(M-1)$. The reduction in diagrams is comparable to reducing a sum over ladders, to a Bethe-Salpeter kernel, although the way it works is, of course, totally different.

In particular, in the equation for the vacuum expectation value of a single Wilson loop, only two types of diagrams survive in the right-hand side. In the first type, every

window is simply connected, and if the left-hand side Wilson loop is rubbed out, the resulting diagram is still connected. In the second type, there is precisely one non-simply-connected window, which looks like a "lake", whose outer perimeter is precisely the left-hand side Wilson loop, and which contains exactly one "island" of propagators.

These equations, called the Group-Variation Equations, are complete, because the Feynman-diagram expansions can be recovered from them order by order in perturbation theory, by developing their solutions in powers of g. They have the advantage, over perturbation theory, that their solutions manifestly have the correct behaviour at large distances, namely the Wilson area law, and massive glueball saturation of the correlation functions.

To observe that this behaviour is a self-consistent solution of the group-variation equations, as discussed briefly above, note that the principal effect of area-law window weights, in the two windows beside a propagator, will be to give that propagator an effective mass, which can be crudely estimated to be at least $1.3\sqrt{\sigma}$, where σ is the area-law parameter. Thus in the sums over paths, paths whose length is greater than $1/\sqrt{\sigma}$ will be suppressed. Thus when the area of the minimal-area spanning surface of the left-hand side Wilson loop is greater than $1/\sigma$, the non-island diagrams in the right-hand side will give a contribution proportional to the perimeter of the loop, whereas the island diagrams will give a contribution proportional to the area of the minimal-area spanning surface of the loop, and thus give the dominant contribution, because we may expect a comparable contribution to the island diagram, from an island of size about $1/\sqrt{\sigma}$, that is situated close enough to any point of the minimal-area spanning surface of the left-hand side loop.

The correlation function of the left-hand side Wilson loop, (of size large compared to $1/\sqrt{\sigma}$), and the Wilson loop defined on the paths that form the outer boundary of the island, will be largest when the island is close to the minimal-area spanning surface S of the left-hand side Wilson loop, and it will approximately factorize into a factor $e^{-\sigma A}$, where A is the area of S, and a factor dependent on the orientation of the island with respect to S, and on the perpendicular distance of the island from S. The result of performing the sums over the island paths, subject to a fixed mean position of the island paths, will be approximately independent of the fixed mean position of the island paths, other than through the overall factor that depends on the perpendicular distance of the island from S.

Thus, for a crude first estimate, we expect that the contribution of any island

diagram, to the right-hand side of the Group-Variation Equation, for the vacuum expectation value of a Wilson loop, whose size is large compared to $1/\sqrt{\sigma}$, will be equal to a constant, times $Ae^{-\sigma A}$, where A is the area of the minimal-area spanning surface, of the left-hand side Wilson loop.

And, as noted briefly above, applying $L\frac{\partial}{\partial L}$, to the left-hand side Wilson loop, also produces a factor of A, thus verifying the consistency of the area law, within a crude first estimate.

A more careful study, in Chapter 7 of the paper, shows that the term $e^{-\sigma A_c}$, in the correlation function of two Wilson loops, where A_c is the area of the cylinder-topology minimal-area orientable spanning surface of the two loops, must develop, when it exists, a pre-exponential factor that decreases, at large A_c, at least as fast as $1/\ln(\sigma A_c)$, because otherwise island diagrams give contributions that are too *large*: in addition to the factor of A, they also get a factor of $\ln(\sigma A)$, which must be cancelled by a pre-exponential factor, that decreases at least as fast as $1/\ln(\sigma A_c)$, at large A_c.

If this pre-exponential factor decreases *faster* than $1/\ln(\sigma A_c)$, at large A_c, then the cylinder-topology minimal-area spanning surface term, in the correlation function of two Wilson loops, gives *no* contribution to the asymptotic form, in the limit of large A, of the right-hand side of the Group-Variation Equation, for the vacuum expectation value of a single Wilson loop, and the entire asymptotic contribution comes from the term, in the correlation function of two loops, that has the form of the lightest glueball, propagating by the shortest possible path, between the separate minimal-area spanning surfaces of the two loops. This term always gives exactly the correct contribution, to the right-hand sides, of the Group-Variation Equations.

As mentioned briefly above, a similar study shows that the island diagrams are also responsible for the self-consistency of the massive glueball saturation of the correlation functions, provided the mass of the lightest glueball is less than twice the effective mass generated for the propagators by the window weights. This is a stringent consistency condition: while it is satisfied for the zeroth-order approximation to the lightest glueball mass, namely $2.38\sqrt{\sigma}$, that results if the pre-exponential factor, in the cylinder-topology minimal-area orientable spanning surface term, in the correlation function of two Wilson loops, decreases *faster* than $1/\ln(\sigma A_c)$, at large A_c, this zeroth-order approximation to the lightest glueball mass is some 33 percent smaller than the best lattice value of $3.56\sqrt{\sigma}$ [4]. Thus more refined estimates, of the effective mass of the propagators, generated by the window weights, will have to give a value of

at least $1.78\sqrt{\sigma}$, in comparison with the simplest estimate, obtained in Chapter 4 of the paper, that the value is at least $1.3\sqrt{\sigma}$. (The nature of the calculation in Chapter 4 shows that the value of $1.3\sqrt{\sigma}$ is conservative, in the sense that it is more likely to be an under-estimate than an over-estimate.)

The island diagram mechanism is robust, in the sense that it works in exactly the same way for the entire sum of island diagrams, as it does for the leading order island diagrams, or for any finite sum of island diagrams. The higher-order island diagrams only give adjustments to the mass of the lightest glueball, as well as adjusting the pre-asymptotic large-distance behaviour of the correlation functions, so that they correspond to a spectrum of massive glueballs with sharp masses.

Furthermore, the island diagram mechanism does not require a particularly large value of g. Indeed, when the renormalization group is used to replace $\frac{\partial}{\partial g}$, in the left-hand side of the group-variation equations, by $L\frac{\partial}{\partial L}$, where L is an overall scaling parameter of the sizes and separations of the Wilson loops involved, the right-hand side becomes multiplied by $\frac{\beta(g)}{g}$. If the right-hand sides are then restricted to the leading island diagrams, in which the islands are simple loops with no vertices, then this factor of $\frac{\beta(g)}{g}$, in the right-hand side, is the *only* explicit dependence on g of the equations: all the remaining dependence on g is through the dependence on g of the vacuum expectation values and correlation functions.

What this means is that, once g reaches a critical value, determined by the group-variation equations, at which the island diagram mechanism operates to produce the correct long-distance behaviour of the vacuum expectation values and correlation functions, there is no need to consider any larger value of g. This is due to the fact that the long-distance behaviour, of all the vacuum expectation values and correlation functions, is completely determined by islands whose size is approximately *fixed*, at about $1/\sqrt{\sigma}$. The critical value of g will have some dependence on the renormalization scheme, but not a lot, since the first two coefficients in $\frac{\beta(g)}{g}$ are independent of the renormalization scheme. Comparison with experimental results suggests that the critical value of $\frac{g^2}{4\pi}$, as normalized in the second half of this paper, will be larger than 0.43, (which corresponds to $\alpha_s(1784\ \text{MeV}) = 0.35$, observed in τ decay [5]), but the most significant point is not the actual value, of the critical value of g, but the fact it is determined by the point, where the absolute value, of $\frac{\beta(g)}{g}$, reaches a certain critical value.

't Hooft has demonstrated, in reference [6], that the sums of the planar Feynman diagrams, in large-N_c QCD, converge geometrically, if one throws away all the diver-

gent subdiagrams, and furthermore, in reference [7], that a similar result holds in the presence of the divergent subdiagrams, if one uses a suitably generalized running coupling, and gives the gluons a mass, to cut off the long-distance growth of the running coupling. The lower bound on the radius of convergence, in the complex g^2 plane, proved by 't Hooft in these references, is several orders of magnitude smaller than the expected critical value of g^2.

It would seem reasonable to make the hypothesis, that in a "natural" renormalization scheme, such as $\overline{\rm MS}$ [8], the convergence behaviour of $\frac{\beta(g)}{g}$, as a power series in g^2, is neither better, nor worse, than the convergence behaviour of other physical quantities, such as the sums in the right-hand sides of the Group-Variation Equations. If this is so, then we may expect the large-N limit of $\frac{\beta(g)}{g}$, as a power series in g^2, to converge geometrically, for sufficiently small g^2. Then since, in $\overline{\rm MS}$, all the coefficients, in the power series for $\frac{\beta(g)}{g}$, seem to have the same sign [5], [9], the fastest direction of growth of $\left|\frac{\beta(g)}{g}\right|$, in the complex g^2 plane, will be along the positive g^2 axis. This implies that the critical value of g^2 will be *strictly smaller* than the radius of convergence of the power series for $\frac{\beta(g)}{g}$, so the power series for $\frac{\beta(g)}{g}$ will *converge geometrically*, at the critical value of g^2.

The same hypothesis then implies that the sums, in the right-hand sides of the Group-Variation Equations, will also *converge geometrically*, at the critical value of g^2.

Study of the large-N limit of the four-loop β-function in $\overline{\rm MS}$, obtained from the general result given in reference [9], shows that the ratios of successive pairs of coefficients in the expansion are increasing, but at a decreasing rate, and indicates that the series is likely to diverge for $\frac{g^2}{4\pi}$, as normalized in the second half of this paper, somewhere in the range 0.53 to 1.05, and most likely, near the lower end of this range. (The four-loop term is essential to reach these conclusions.) The critical value of $\frac{g^2}{4\pi}$ is expected to be strictly smaller than the value where the series diverges.

But as noted earlier, the experimental result, from observations of τ decay, that $\alpha_s(1748\ {\rm MeV}) = 0.35$, indicates that the critical value, of this $\frac{g^2}{4\pi}$, is greater than 0.43. This is because this $\frac{g^2}{4\pi}$ is equal to $\frac{3}{2}$ times the value α_s would have in the absence of quarks, which at 1748 MeV is approximately $\frac{3}{2} \times \frac{1}{1.22} \times \alpha_s = 1.23\alpha_s$. Thus $\alpha_s(m_\tau)$, which is the largest value of α_s for which there is experimental evidence, must be very close to the critical value.

It is interesting to note, in Figure 9.2, on page 19, of Chapter 9, *Quantum Chromodynamics*, of reference [5], that the experimental value, of $\alpha_s(m_\tau)$, lies about 1.6

standard deviations above the best fit curve, to all measurements, of α_s. This suggests that the curve of $\alpha_s(\mu)$ is indeed starting to curve upwards towards a vertical slope, as μ approaches m_τ from above, which is what is expected as $\frac{\beta(g)}{g}$ starts to diverge.

Since the experimental value of $\sqrt{\sigma}$ is about 0.44 GeV [10], this experimental evidence, that α_s is approaching the critical value as μ approaches 1748 MeV $= 4.0\sqrt{\sigma}$, is a further indication that the estimate in Chapter 4 of this paper, that the effective mass generated for the propagators by the window weights, is at least $1.3\sqrt{\sigma}$, is an underestimate. Indeed, the typical size, of the islands, of approximately fixed size, that determine the long distance behaviour of all the vacuum expectation values and correlation functions, is approximately equal to the reciprocal, of twice the effective mass generated for the propagators by the window weights, since when an island elongates in any direction, at least two of the propagators in the island get elongated. Thus since the mass at which α_s stops evolving, is equal to the reciprocal of the size of these typical islands, the mass at which α_s stops evolving, is equal to about twice the effective mass generated for the propagators, by the window weights. Thus the experimental evidence that α_s is stopping evolving, by reaching the critical value, at about $\mu = 4.0\sqrt{\sigma}$, indicates that the effective mass, generated for the propagators by the window weights, is about $2.0\sqrt{\sigma}$, rather than $1.3\sqrt{\sigma}$.

Since the long-distance behaviour of vacuum expectation values and correlation functions is determined, via the island diagram mechanism, by islands whose size is approximately fixed at $\frac{1}{4.0\sqrt{\sigma}}$, and this mechanism works in exactly the same way for the full sum of island diagrams, as it does for the leading-order island diagrams, or the sum of any finite number of island diagrams, there is no physical reason why the sums, in the right-hand sides of the Group-Variation Equations, should not converge.

There is also a possibility that the Group-Variation Equations might be systematically solvable, by the iterative substitution of their left-hand sides into their right-hand sides, starting from a reasonable ansatz.

Whether the $1/N$ expansions themselves can converge, is more dubious. For example, each time N decreases from a positive integer, to the next smaller positive integer, some of the states must drop out of the spectrum. Whether this occurs by their masses becoming infinite, or by their coefficients, in the expansion of any correlation function in terms of "eigenstates", somehow vanishing for all integer values of N below a certain integer, or by some mechanism connected with the fact that most of the states are also developing widths, is not known.

The fact that, within the context of the Group-Variation Equations, α_s does not evolve, at distances larger than the typical island size, might be compared with the fact that, within quantum electrodynamics, the fine structure constant does not evolve, at distances larger than the reciprocal of the mass of the lightest charged particle, namely the electron.

The foregoing introduction has omitted many important points, and some difficulties, that are considered in detail in the paper. The general idea of a "two-stage" integration over the gauge fields of some large gauge group, (which in practice will be $\mathrm{SU}(NM)$), in which the gauge fields that are *not* members of some subgroup, (which in practice will be $(\mathrm{SU}(N))^M$), are integrated over first, treating the gauge fields in the subgroup as background fields, after which the gauge fields in the subgroup are integrated over, enabling the vacuum expectation values and correlation functions, of gauge-invariant quantities in the larger group, to be expressed in terms of sums over diagrams, involving sums over paths, weighted by the vacuum expectation values and correlation functions, of gauge-invariant quantities in the smaller group, is most simply studied in the context of a general larger group H, and subgroup G, resulting in equations called the Group-Changing Equations for H and G. A special choice of gauge-fixing and Fadeev-Popov terms must be made, which is, however, renormalizable, and such that the sum of the gauge-fixing and Fadeev-Popov terms is a BRST variation in the normal way, with completely standard BRST variations. There is a normal gauge parameter, but the requirement that Fadeev-Popov loops stay either in the subgroup or out of it, rather than wandering in and out of the subgroup, requires that the analogue of Landau gauge must be chosen.

Starting from this systematic basis, the group-variation equations can be derived, not only for the leading non-vanishing terms in the $1/N$ expansions, but for all the terms in the $1/N$ expansions. The coefficient of every term in, the $1/N$ expansion, of the vacuum expectation value or correlation function, of a product of Wilson loops, has its own Group-Variation Equation, and the Group-Variation Equations for the n^{th} non-vanishing terms, in the vacuum expectation value of one Wilson loop, and the correlation functions of two or more Wilson loops, close among themselves, and are complete, in the sense that the Feynman-diagram expansions can be recovered, by developing them, as power series, in the coupling constant.

Explicitly, the equations for the leading non-vanishing term, in the vacuum expectation value, of one Wilson loop, and the leading non-vanishing terms, in the correlation

functions, of two or more Wilson loops, are complete, and close among themselves, the equations for the next-to-leading non-vanishing term, in the vacuum expectation value, of one Wilson loop, and the next-to-leading non-vanishing terms, in the correlation functions, of two or more Wilson loops, are complete, and close among themselves, but with the leading non-vanishing terms also in the right-hand sides, and so on. Thus the equations for the leading non-vanishing contributions can be solved, then these solutions can be used, in the right-hand sides, of the equations for the next-to-leading non-vanishing contributions, and so on. Thus the Group-Variation Equations can be used to calculate all the Yang Mills vacuum expectation values, and correlation functions, required for the study of mesons and glueballs. [3] Witten has given evidence, in reference [11], that baryons are monopoles, or solitons, of large-N_c QCD, to be studied in a Hartree-Fock manner. It remains to be determined whether the Group-Variation Equations can be used to calculate the Yang Mills vacuum expectation values, and correlation functions, required for the study of baryons. Some recent results, in the application of the $1/N_c$ expansion to baryons, are given in references [12], [13], [14], and [15].

The determination of the form of the group-variation equations to all orders in $1/N$, and in particular, determining what happens to the linear combinations of the diagonal elements of an SU(NM) matrix, that are *not* elements of the $(\mathrm{SU}(N))^M$ subgroup, requires choosing a specific basis for the generators of SU(NM), that is suited to the $(\mathrm{SU}(N))^M$ subgroup. In the basis used in this paper, the off-diagonal SU(NM) generators, that are not members of the $(\mathrm{SU}(N))^M$ subgroup, are called t_4's and t_5's, and the diagonal SU(NM) generators, that are not members of the block-diagonal $(\mathrm{SU}(N))^M$ subgroup, are called t_6's. The "6-fields" do not couple at all to the fields in the $(\mathrm{SU}(N))^M$ subgroup, and thus have free propagators, even in the presence of background fields in the $(\mathrm{SU}(N))^M$ subgroup. They have no occurrence at all, in the Group-Variation Equations for the leading non-vanishing terms, in the vacuum expectation values and correlation functions.

Determining the detailed form of the Group-Variation Equations, and in particular, verifying that the equations for the leading non-vanishing terms, in the $1/N$ expansions, have the expected simple form, involves identifying some eight "selection rules," that restrict the diagrams that can occur, in the Group-Changing Equations, for SU(NM)

[3] The determination of the form of the Group-Variation Equations, to all orders in $1/N$, was carried out in response to a question from G. Ross.

and $(\mathrm{SU}(N))^M$.

As mentioned briefly above, the detailed study, of the island diagram mechanism, shows that the introductory description, given above, must be modified for the following reason. If the correlation function, of a Wilson loop of size approximately $1/\sqrt{\sigma}$, close to the minimal-area spanning surface of a much larger Wilson loop, is proportional to $e^{-\sigma A_c}$, with no pre-exponential factor, where A_c is the area of the *cylinder-topology* minimal-area spanning surface, whose boundary is the two loops, properly oriented, then the contribution of the island diagrams is *too large*: it grows in proportion to $L^2 \ln L$, as the dimensions of the left-hand side Wilson loop are scaled by a factor L, rather than in proportion to L^2, as required. This would require the vacuum expectation value, of the left-hand side Wilson loop, to depend on L as $ae^{-bL^2 \ln(cL)}$, which is impossible, because it violates the Seiler bound [16][17].

The simplest resolution, of this problem, is that the cylinder-topology minimal-area spanning surface term, in the correlation function of two Wilson loops, must be multiplied, when it exists, by a pre-exponential factor, that decreases at least as fast as $\frac{1}{\ln(\sigma A_c)}$, at large A_c.

This behaviour occurs because, if the area of the minimal-area spanning surface of the small loop is $1/\sigma$, and the area of the minimal-area spanning surface, S, of the large loop, is A, then in the limit of large σA, if the perpendicular distance, z, from the small loop, to S, is not too large, and the perpendicular projection, of the small loop, onto S, is not too near the edge of S, or in other words, not too near the large loop itself, and the small loop is oriented, so as to minimize the area, of the cylinder-topology, minimal-area, spanning surface, S_c, of the two loops, then the area, A_c, of S_c, depends on the perpendicular distance, z, from the small loop to S, only through the additive term $\frac{2\pi z^2}{\ln(\sigma A)}$. The $\ln(\sigma A)$, in the denominator, means that the larger the area A, of S, the more weakly the small loop is attracted to S. It is this denominator factor, of $\ln(\sigma A)$, that produces the unacceptable factor of $\ln(L)$, in the island diagram contributions, if the cylinder-topology term, in the correlation function of two Wilson loops, behaves, when it exists, simply as $e^{-\sigma A_c}$, with no pre-exponential factor.

As mentioned briefly above, if the pre-exponential factor, in the cylinder-topology term, in the correlation function of two Wilson loops, decreases *faster* than $\frac{1}{\ln(\sigma A_c)}$, at large A_c, then this term makes *no* contribution to the asymptotic form, at large A, of the right-hand side, of the Group-Variation Equation, for the vacuum expectation value, of a Wilson loop, the area of whose minimal-area spanning surface, is A. If

this is the case, then the asymptotic form, at large A, of the right-hand side of this Group-Variation Equation, comes entirely from the term $f^2\sqrt{\frac{m}{32\pi^3 z^3}}e^{-mz}e^{-\sigma A_1}e^{-\sigma A_2}$, in the correlation function of two Wilson loops, whose separate minimal-area spanning surfaces, S_1, and S_2, have areas A_1, and A_2, where z is the shortest distance between any point of S_1, and any point of S_2, m is the mass of the lightest glueball, and f is the glueball-to-surface coupling constant. This immediately gives the correct form of the Wilson area law, and also, in the approximation that the sums, over the island diagrams, are dominated by their leading terms, gives the zeroth-order approximation, $m = 2.38\sqrt{\sigma}$, for the mass of the lightest glueball, which is about 33 percent less than the best lattice value, of $3.56\sqrt{\sigma}$ [4].

In Section 8.1, I give some evidence that the Group-Variation Equations can be regularized by dimensional regularization [18], [19], [20], in a gauge-invariant manner, which also preserves the property that Fadeev-Popov loops stay either in the subgroup or out of it, and in Section 8.2, I give some evidence that this can also by achieved, by the method of adding gauge-invariant higher derivative terms to the action, plus Pauli-Villars scalar and spinor regulator fields, to cancel the one-loop divergences [21].

The Group Variation Equations represent a minimum resummation of the Feynman diagrams contributing to the coefficients, in the 1/N expansions, of physical quantities in SU(N) gauge theories, in that each side, of a Group Variation Equation, is simply equal to $g^2\frac{\partial}{\partial g^2}$, plus a finite integer constant, (1 or 0 in the simplest cases), acting on the corresponding sum of Feynman diagrams.

Shortly after the discovery of asymptotic freedom, Gross and Wilczek [22] suggested that the endless increase of the coupling constant, at long distances, might be responsible for quark confinement, and Weinberg [23] suggested an explanation based on infra-red divergences, and the masslessness of the gluons, while 't Hooft, in references [6] and [7], later showed that neither of these effects, in the absence of the other, can produce confinement, within the direct sum of planar Feynman diagrams. It has turned out that all the coefficients in $\beta(g)$ appear to have the same sign, so that $\left|\frac{\beta(g)}{g}\right|$ runs to infinity at a finite distance, less than $\frac{1}{\Lambda_s} \simeq \frac{1}{200\text{MeV}}$, so that perturbation theory cannot be used at long distances. In this paper we see that the two effects together produce confinement by the island diagram mechanism, and that within the Group-Variation Equations, the window weights produce an effective mass for the gluon paths, that cuts off the infra-red divergences, and ensures that the long-distance behaviour, of vacuum expectation values and correlation functions, is determined by islands of fixed

size $\simeq \frac{1}{4.0\sqrt{\sigma}} \simeq \frac{1}{1800\text{MeV}}$, so that the running coupling never increases beyond a critical value $\alpha_s \simeq 0.35$, at which the sums in the right-hand sides of the Group-Variation Equations probably converge, so that a minimal resummation of perturbation theory, as given by the Group-Variation Equations, can be used for the calculation of all strong interaction quantities, that do not involve baryons.

Chapter 1

Group-Changing Equations

1.1 Background and Conventions

We consider Yang Mills theory [27] for a compact Lie algebra G in 4 Euclidean dimensions, with the conventions that the matrices t_a of a representation of G satisfy $(t_a)^\dagger = -t_a$, and $[t_a, t_b] = f_{abc}t_c$, where f_{abc} are the real and totally antisymmetric structure constants, (so that the adjoint representation of G is given by $(t_c)_{ab} = f_{acb}$), and that the gauge variation of the gauge field $A_{\mu a}$ is $A_{\mu a} \to A_{\mu a} + \partial_\mu \epsilon_a + A_{\mu b} f_{abc} \epsilon_c$, so that the general covariant derivative is $D_{\mu ij}\psi_j = \partial_\mu \psi_i + A_{\mu b}(t_b)_{ij}\psi_j$, and the gauge variation of the matter field ψ_i is $\psi_i \to \psi_i - \epsilon_c(t_c)_{ij}\psi_j$. The Yang Mills action density is $\frac{1}{4g^2}F_{\mu\nu a}F_{\mu\nu a}$, where $F_{\mu\nu a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} + f_{abc}A_{\mu b}A_{\nu c}$, and g is the coupling constant.

We wish to determine the vacuum expectation values and correlation functions of gauge-invariant quantities constructed from the gauge-invariant path-ordered phase-factor:

$$W(A, x(s))_{ij} = \sum_{n=0}^{\infty} \int_0^1 ds_1 \ldots \int_0^1 ds_n \theta(s_2 - s_1) \ldots \theta(s_n - s_{n-1}) \frac{dx_{\mu_1}(s_1)}{ds_1} \ldots \frac{dx_{\mu_n}(s_n)}{ds_n} \quad \times$$
$$\times \quad A_{\mu_1 a_1}(x(s_1)) \ldots A_{\mu_n a_n}(x(s_n))(t_{a_1} \ldots t_{a_n})_{ij} \tag{1.1}$$

where the continuous path $x(s)$, $0 \le s \le 1$, either consists of a finite number of straight segments, or else is a smooth curve, and $\theta(s)$ is the step function, $\theta(s) = 1$ for $s \ge 0$, $\theta(s) = 0$ for $s < 0$.

The gauge-variation of W is given by:

$$\begin{aligned} W(A + D\epsilon, x(x))_{ij} &= W(A, x(s))_{ij} - \epsilon_b(x(0))(t_b)_{ik} W(A, x(s))_{kj} + \\ &\quad + W(A, x(s))_{ik} \epsilon_b(x(1))(t_b)_{kj} \end{aligned} \tag{1.2}$$

The simplest form of gauge-invariant quantity formed from W is obtained by taking the trace of W when $x(s)$ is a closed path, so that $x(0)$ is equal to $x(1)$. This is called a Wilson loop. More general gauge-invariant quantities may be formed from the W's, possibly in different representations, by taking a network of paths meeting at junctions, and contracting the W's at the junctions with invariant tensors whose indices are in the appropriate representations.

1.2 Gauge Fixing and BRS Invariance

We consider a compact Lie algebra H and its compact Lie sub-algebra G, and obtain integral equations expressing the vacuum expectation values, in the Yang Mills theory for H, of the gauge-invariant quantities of H, in terms of the vacuum expectation values, in the Yang Mills theory for G, of the gauge-invariant quantities of G. We let lower case letters a, b, c, ... run over the elements of the Lie algebra G, and upper-case letters A, B, C, ... run over the elements of the Lie algebra H that are *not* elements of G, and we use the summation convention that repeated indices run over the domains just defined. Then the fact that G is a sub-*algebra* of H is expressed by the vanishing of the structure constants f_{abC} with two lower-case indices and one upper-case index, so that there is no $f_{abC}t_C$ term in the right-hand side of the commutation relation $[t_a, t_b] = f_{abc}t_c$. Now with Greek indices running over the *whole* of H, the Jacobi identity for H may be written:

$$f_{\gamma\alpha\epsilon}f_{\epsilon\beta\delta} - f_{\gamma\beta\epsilon}f_{\epsilon\alpha\delta} = f_{\alpha\beta\epsilon}f_{\gamma\epsilon\delta} \tag{1.3}$$

and taking γ as C, β as b, and δ as D in this equation, and noting that f_{Cae} and f_{abE} are equal to zero, we obtain:

$$f_{CaE}f_{EbD} - f_{CbE}f_{EaD} = f_{abe}f_{CeD} \tag{1.4}$$

Thus the matrices $(t_a)_{CD} = f_{CaD}$ form a representation, (possibly reducible), of the Lie algebra G.

We denote the H-covariant derivative by D_μ, and the G-covariant derivative by $\bar{D}_\mu$. The fields $A_{\mu A}$ transform as matter fields under gauge variations in G, and their G-covariant derivative is given by:

$$(\bar{D}_\mu A_\nu)_A = \partial_\mu A_{\nu A} + A_{\mu a}f_{AaB}A_{\nu B} \tag{1.5}$$

We introduce gauge-fixing auxiliary fields B_a and B_A, and choose the gauge-fixing action to be:

$$\frac{1}{g^2}\left(iB_a(\partial_\mu A_{\mu a}) + \frac{\alpha}{2}B_a B_a + iB_A(\bar{D}_\mu A_\mu)_A + \frac{\beta}{2}B_A B_A\right) \tag{1.6}$$

We then find in the usual way that we must add the Fadeev-Popov action density:

$$\frac{1}{g^2}\left(\psi_a(\partial_\mu(D_\mu\phi)_a) + \psi_A(\bar{D}_\mu(D_\mu\phi)_A) + \psi_A(D_\mu\phi)_a f_{AaB}A_{\mu B}\right) \tag{1.7}$$

where ψ_a, ϕ_a, ψ_A, and ϕ_A are the Fadeev-Popov fields.

Now if we again allow Greek indices α, β, γ, ... to run over the *whole* of H, and indicate position arguments by subscripts x, y, ..., then the standard BRS operator [27] δ for H may be written:

$$\delta = \int d^4x \left((D_\mu\phi)_{\alpha x}\frac{\delta}{\delta A_{\mu\alpha x}} - \frac{1}{2}f_{\alpha\beta\gamma}\phi_{\beta x}\phi_{\gamma x}\frac{\delta}{\delta\phi_{\alpha x}} + iB_{\alpha x}\frac{\delta}{\delta\psi_{\alpha x}}\right) \tag{1.8}$$

We note that, in consequence of our use of the gauge-fixing auxiliary fields B_α, δ^2 vanishes *exactly*, $\delta^2 = 0$.

Now the sum of our gauge-fixing and Fadeev-Popov action densities (1.6) and (1.7) is equal to the action of δ on:

$$\frac{1}{g^2}\left(\psi_a(\partial_\mu A_{\mu a}) - i\frac{\alpha}{2}\psi_a B_a + \psi_A(\bar{D}_\mu A_\mu)_A - i\frac{\beta}{2}\psi_A B_A\right) \tag{1.9}$$

Hence it directly follows from the nilpotence of δ that our full action has the standard BRS invariance for H, hence that the effective action Γ, in the presence of sources for $(\bar{D}_\mu A_\mu)_A$ and for the BRS variations of $A_{\mu\alpha}$ and ϕ_α, satisfies Ward identities of the usual form [28].

Indeed, if this structure can be preserved by a gauge-invariant regularization, we could conclude directly, from the general result, that two actions which differ by a BRS-variation, give the same results for the vacuum expectation values of BRS-invariant quantities, that this gauge-fixing procedure will give the same results for the vacuum expectation values of BRS-invariant quantities, as the usual one.

Now g^2 times the Fadeev-Popov action density (1.7) is equal to:

$$\left(\psi_a(\partial^2\phi_a + \partial_\mu(A_{\mu b}f_{abc}\phi_c) + \partial_\mu(A_{\mu B}f_{aBC}\phi_C)) + \right.$$

$$+\psi_A(\partial^2\phi_A + \partial_\mu(A_{\mu b}f_{AbC}\phi_C) + \partial_\mu(A_{\mu B}f_{ABc}\phi_c) + \partial_\mu(A_{\mu B}f_{ABC}\phi_C)) +$$

$$+\psi_A A_{\mu a} f_{AaE}(\partial_\mu \phi_E + A_{\mu b} f_{EbC}\phi_C + A_{\mu B} f_{EBc}\phi_c + A_{\mu B} f_{EBC}\phi_C) +$$
$$+ \psi_A(\partial_\mu \phi_a + A_{\mu c} f_{acd}\phi_d + A_{\mu C} f_{aCD}\phi_D) f_{AaB} A_{\mu B}\Big) \tag{1.10}$$

We examine the terms that contain both ψ with an *upper*-case index and ϕ with a *lower*-case index. The first such term is $\psi_A(\partial_\mu(A_{\mu B} f_{ABc}\phi_c))$. The second such term is $\psi_A A_{\mu a} f_{AaE}(A_{\mu B} f_{EBc}\phi_c)$, which is equal to $-\psi_A A_{\mu c}\phi_d A_{\mu B} f_{AcE} f_{EdB}$. The third such term is $\psi_A(\partial_\mu \phi_a) f_{AaB} A_{\mu B}$, which is equal to $-\psi_A A_{\mu B} f_{ABc}(\partial_\mu \phi_c)$. And the fourth and final such term is $\psi_A A_{\mu c} f_{acd}\phi_d f_{AaB} A_{\mu B}$, which by (1.4) is equal to $\psi_A A_{\mu c}\phi_d A_{\mu B}(f_{AcE} f_{EdB} - f_{AdE} f_{EcB})$. Thus the sum of the first and third such terms is equal to $\psi_A(\partial_\mu A_{\mu B}) f_{ABc}\phi_c$, and the sum of the second and fourth such terms is equal to:

$$-\psi_A A_{\mu c}\phi_d A_{\mu B} f_{AdE} f_{EcB} = \psi_A(A_{\mu c} f_{EcB} A_{\mu B}) f_{AEd}\phi_d = \psi_A(A_{\mu a} f_{BaC} A_{\mu C}) f_{ABc}\phi_c$$

Hence the sum of all four such terms is equal to:

$$\psi_A(\bar{D}_\mu A_\mu)_B f_{ABc}\phi_c \tag{1.11}$$

1.3 Propagators for Fields not in the Subgroup, in the presence of Background Fields in the Subgroup

We now define $\bar{F}_{\mu\nu a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} + f_{abc} A_{\mu b} A_{\nu c}$, so that $F_{\mu\nu a}$ is equal to $\bar{F}_{\mu\nu a} + f_{aBC} A_{\mu B} A_{\nu C}$, and we note that $F_{\mu\nu A}$ is equal to $(\bar{D}_\mu A_\nu)_A - (\bar{D}_\nu A_\mu)_A + f_{ABC} A_{\mu B} A_{\nu C}$. Thus there are *no* terms in $F_{\mu\nu a} F_{\mu\nu a} + F_{\mu\nu A} F_{\mu\nu A}$ that contain exactly one A_μ with an upper-case group index, and the sum of all the terms in $F_{\mu\nu a} F_{\mu\nu a} + F_{\mu\nu A} F_{\mu\nu A}$ that contain exactly two A_μ's with upper-case group indices, is:

$$2((\bar{D}_\mu A_\nu)_A(\bar{D}_\mu A_\nu)_A - (\bar{D}_\nu A_\mu)_A(\bar{D}_\mu A_\nu)_A + \bar{F}_{\mu\nu a} f_{aBC} A_{\mu B} A_{\nu C}) \tag{1.12}$$

Thus if we consider $A_{\mu A}$ and B_A to be propagating in a *background* field given by $A_{\mu a}$, and if we again denote position arguments by subscripts x, y, ..., and if we define the propagator matrix for $A_{\mu A}$ and B_A propagating in the background field $A_{\mu a}$ by:

$$(A_{\mu Ax}, B_{Ax}) \begin{pmatrix} G_{\mu Ax,\nu By} & \tilde{G}_{\mu Ax,By} \\ G_{Ax,\nu By} & G_{Ax,By} \end{pmatrix} \begin{pmatrix} A_{\nu By} \\ B_{By} \end{pmatrix} \tag{1.13}$$

then we find that this propagator matrix satisfies the equation:

$$\left(\begin{array}{c|c} -(\bar{D}^2)_{ACx}\delta_{\mu\sigma} + (\bar{D}_\mu\bar{D}_\sigma)_{ACx} - 2\bar{F}_{\mu\sigma ax}f_{AaC} & -i(\bar{D}_\mu)_{ACx} \\ i(\bar{D}_\sigma)_{ACx} & \beta\delta_{AC} \end{array}\right)\left(\begin{array}{cc} G_{\sigma Cx,\nu By} & \tilde{G}_{\sigma Cx,By} \\ G_{Cx,\nu By} & G_{Cx,By} \end{array}\right) =$$

$$= g^2\left(\begin{array}{c|c} \delta_{\mu\nu}\delta_{AB}\delta^4(x-y) & 0 \\ 0 & \delta_{AB}\delta^4(x-y) \end{array}\right) \tag{1.14}$$

which we abbreviate as:

$$\left(\begin{array}{c|c} -\bar{D}^2\delta_{\mu\sigma} + \bar{D}_\mu\bar{D}_\sigma - 2\bar{F}_{\mu\sigma} & -i\bar{D}_\mu \\ i\bar{D}_\sigma & \beta \end{array}\right)\left(\begin{array}{cc} G_{\sigma\nu} & \tilde{G}_\sigma \\ G_\nu & G \end{array}\right) = g^2\left(\begin{array}{cc} \delta_{\mu\nu} & 0 \\ 0 & 1 \end{array}\right) \tag{1.15}$$

We define the operator E by:

$$E = \bar{D}_\mu\frac{1}{\bar{D}^2}\bar{D}_\mu \tag{1.16}$$

and we note that $E_{Ax,By}$ is equal to $\delta_{AB}\delta^4(x-y)$ plus terms of degree one and higher in $A_{\mu a}$.

Now the exact solution of the equation obtained from (1.15) by deleting the term $-2\bar{F}_{\mu\sigma}$, is given by:

$$g^2\left(\begin{array}{c|c} -\frac{1}{\bar{D}^2}\delta_{\sigma\nu} + \frac{1}{\bar{D}^2}\bar{D}_\sigma\left(\frac{1-\beta}{(1-\beta)E+\beta}\right)\bar{D}_\nu\frac{1}{\bar{D}^2} & \frac{1}{\bar{D}^2}\bar{D}_\sigma\left(\frac{-i}{(1-\beta)E+\beta}\right) \\ \left(\frac{i}{(1-\beta)E+\beta}\right)\bar{D}_\nu\frac{1}{\bar{D}^2} & \left(\frac{1-E}{(1-\beta)E+\beta}\right) \end{array}\right) \tag{1.17}$$

And if we denote (1.17) by g^2G_0, and the exact solution of (1.15) by g^2G_1, and the matrix $\left(\begin{array}{cc} 2\bar{F}_{\mu\sigma} & 0 \\ 0 & 0 \end{array}\right)$ by F, then the exact solution of (1.15) is given by:

$$g^2G_1 = g^2\left(G_0 + G_0FG_0 + G_0FG_0FG_0 + G_0FG_0FG_0FG_0 + \ldots\right) \tag{1.18}$$

And furthermore, if we denote the AA component of (1.17) by g^2G_0, and the AA component of the exact solution of (1.15) by g^2G_1, then in consequence of the vanishing of the AB, BA, and BB components of F, the AA component of the exact solution of (1.15) is also given by (1.18), with F now interpreted simply as $\bar{F}_{\mu\sigma}$.

1.3.1 Landau gauge

We observe that, for $\beta \neq 0$, it follows from (1.17) and (1.18) that the AA component of the exact solution of (1.15) is also the exact solution of the equation for $G_{\mu Ax,\nu By}$

alone that is obtained if the gauge-fixing auxiliary field has been integrated out:

$$\left(-(\bar{D}^2)_{ACx}\delta_{\mu\sigma}+\left(1-\frac{1}{\beta}\right)(\bar{D}_\mu\bar{D}_\sigma)_{ACx}-2\bar{F}_{\mu\sigma ax}f_{aAC}\right)G_{\sigma Cx,\nu By}=g^2\delta_{\mu\nu}\delta_{AB}\delta^4(x-y) \tag{1.19}$$

And we observe that, as β tends to 0, the propagator matrix (1.17) tends smoothly to the Landau gauge form:

$$g^2\left(\begin{array}{c|c} -\frac{1}{\bar{D}^2}\delta_\sigma\nu+\frac{1}{\bar{D}^2}\bar{D}_\sigma\frac{1}{E}\bar{D}_\nu\frac{1}{\bar{D}^2} & -i\frac{1}{\bar{D}^2}\bar{D}_\sigma\frac{1}{E} \\ i\frac{1}{E}\bar{D}_\nu\frac{1}{\bar{D}^2} & \frac{1}{E}-1 \end{array}\right) \tag{1.20}$$

Thus we see that the Landau gauge case $\beta = 0$ may be treated without any problems, without any need for any limiting process from $\beta \neq 0$, and without any reference at all to the gauges with $\beta \neq 0$, by means of the gauge-fixing auxiliary field B_A, and that the results obtained by this method are in exact agreement with the results obtained by letting β tend to 0 in the results obtained, for general $\beta \neq 0$, without the use of the gauge-fixing auxiliary field.

We note furthermore [31] that the Landau gauge condition $\beta = 0$ is preserved under changes of renormalization point, so that in Landau gauge there is no $\frac{\partial}{\partial\beta}$ term in the renormalization group equation, even for non-gauge-invariant quantities.

1.3.2 Fadeev-Popov loops stay either in or out of the subgroup in Landau gauge

Now the AA component of (1.20) satisfies the identity:

$$\bar{D}_\sigma\left(-\frac{1}{\bar{D}^2}\delta_{\sigma\nu}+\frac{1}{\bar{D}^2}\bar{D}_\sigma\frac{1}{E}\bar{D}_\nu\frac{1}{\bar{D}^2}\right)=0 \tag{1.21}$$

Hence it immediately follows from (1.18) that in Landau gauge $G_{\mu Ax,\nu By}$, which is the exact $A_{\mu Ax}A_{\nu By}$ propagator in the background field $A_{\mu a}$, satisfies the indentity:

$$\bar{D}_\mu G_{\mu\nu}=0 \tag{1.22}$$

Hence it immediately follows from the form of (1.11) that in Landau gauge for $A_{\mu A}$, that is, for $\beta = 0$ in (1.6), and for the vacuum expectation value of any quantity that includes no B_A's, the terms in the Fadeev-Popov action density (1.10) that contain both ψ with an *upper*-case index and ϕ with a *lower*-case index, make *no contribution at all*. Furthermore, in the vacuum expectation value of any quantity that includes

no Fadeev-Popov fields, the Fadeev-Popov propagators occur only in closed loops, and, since the Fadeev-Popov propagators do not mix upper-case indices and lower-case indices, any Fadeev-Popov loop that includes any vertex that contains both ψ with a *lower*-case index and ϕ with an *upper*-case index, must also include at least one vertex that contains both ψ with an *upper*-case index and ϕ with a *lower*-case index. Hence if we use Landau gauge for $A_{\mu A}$, or in other words, if we set $\beta = 0$ in (1.6), and if we are calculating the vacuum expectation value of any quantity that includes no B_A's and no Fadeev-Popov fields, then we may completely neglect the five terms in the Fadeev-Popov action density (1.10) that contain both a Fadeev-Popov field with an upper-case index and a Fadeev-Popov field with a lower-case index. The remaining terms in (1.10) consist of the standard Fadeev-Popov action density $\psi_a(\partial^2\phi_a + \partial_\mu(A_{\mu b}f_{abc}\phi_c))$ for G, plus seven terms that may be put in the manifestly G-gauge-invariant form:

$$\psi_A(\bar{D}^2\phi)_A + \psi_A(\bar{D}_{\mu AE}(A_{\mu B}f_{EBC}\phi_C)) + \psi_A f_{AaB}A_{\mu B}\phi_C f_{CaD}A_{\mu D} \tag{1.23}$$

1.3.3 The propagators expressed as sums over paths, weighted by Wilson lines of fields in the subgroup

We next note that $\left(\frac{1}{\bar{D}^2}\right)_{Ax,By}$ may be expressed as a sum over paths from x to y, each weighted by the path-ordered phase factor (1.1), in the representation of G given by $(t_a)_{CD} = f_{CaD}$. Indeed, we may write:

$$\left(\frac{-1}{\bar{D}^2}\right)_{Ax,By} = \int_0^\infty ds \left(e^{s\bar{D}^2}\right)_{Ax,By} \tag{1.24}$$

We choose a value σ, which is to represent the "maximum tolerable" width of a Gaussian, and write:

$$\int_0^\infty ds \left(e^{s\bar{D}^2}\right)_{Ax,By} = \sum_{n=0}^\infty \int_{n\sigma}^{(n+1)\sigma} ds \left(e^{s\bar{D}^2}\right)_{Ax,By} \tag{1.25}$$

Then for $n \geq 1$, with $n\sigma \leq s \leq (n+1)\sigma$, we write:

$$\left(e^{s\bar{D}^2}\right)_{Ax,By} = \int d^4z_1 \ldots \int d^4z_n \left(e^{\frac{s\bar{D}^2}{n+1}}\right)_{Ax,C_1z_1} \left(e^{\frac{s\bar{D}^2}{n+1}}\right)_{C_1z_1,C_2z_2} \ldots \left(e^{\frac{s\bar{D}^2}{n+1}}\right)_{C_nz_n,By} \tag{1.26}$$

Then in this exact expression, noting that $\frac{n\sigma}{n+1} \leq \frac{s}{n+1} \leq \sigma$ holds, we approximate each $\left(e^{\frac{s\bar{D}^2}{n+1}}\right)_{Ep,Fq}$ by $\left(e^{\sigma\partial^2}\right)_{p,q} = \frac{e^{-\frac{(p-q)^2}{4\sigma}}}{(4\pi\sigma)^2}$ multiplied by the path-ordered phase factor $W_{Ep,Fq}$ for the *straight line* from p to q.

Thus if we denote by $(W_{xz_1z_2\ldots z_ny})_{AB}$ the path-ordered phase factor for the path consisting of the straight line from x to z_1, then the straight line from z_1 to z_2, and so on, and then finally the straight line from z_n to y, then the σ-approximation to $\left(\frac{-1}{D^2}\right)_{Ax,By}$ is given by:

$$\sigma\sum_{n=0}^{\infty}\int d^4z_1\ldots\int d^4z_n\,(W_{xz_1z_2\ldots z_ny})_{AB}\,\frac{e^{-\frac{(x-z_1)^2}{4\sigma}}}{(4\pi\sigma)^2}\frac{e^{-\frac{(z_1-z_2)^2}{4\sigma}}}{(4\pi\sigma)^2}\ldots\frac{e^{-\frac{(z_n-y)^2}{4\sigma}}}{(4\pi\sigma)^2} \tag{1.27}$$

In Section 8.3 we sketch the derivation, directly from the $\sigma\to 0$ limit of this expression, of the standard expansion:

$$\frac{-1}{\bar{D}^2}=\frac{-1}{\partial^2}+\frac{-1}{\partial^2}(\partial A+A\partial+AA)\frac{-1}{\partial^2}+\frac{-1}{\partial^2}(\partial A+A\partial+AA)\frac{-1}{\partial^2}(\partial A+A\partial+AA)\frac{-1}{\partial^2}+\ldots \tag{1.28}$$

where $\left(\frac{-1}{\partial^2}\right)_{xy}=\frac{1}{4\pi^2(x-y)^2}$.

The G-covariant derivative, acting on $A_{\nu Ax}$, for example, may be approximated by:

$$(\bar{D}_\mu A_\nu)_{Ax}\simeq\int d^4y\frac{(y-x)_\mu}{2\sigma}\frac{e^{-\frac{(y-x)^2}{4\sigma}}}{(4\pi\sigma)^2}W_{Ax,By}A_{\nu By} \tag{1.29}$$

Now as noted after equation (1.16), $E_{Ax,By}$ is equal to $\delta_{AB}\delta^4(x-y)$ plus terms of degree one and higher in $A_{\mu a}$. Thus $\frac{1}{E}$ may be expressed as:

$$\frac{1}{E}=\frac{1}{1-(1-E)}=1+(1-E)+(1-E)^2+\ldots \tag{1.30}$$

Hence, by (1.16), (1.27), and (1.29), $\frac{1}{E}$ may also be expressed in terms of sums over paths, weighted by the path-ordered phase factor. Hence both the Landau-gauge $A_{\mu A}A_{\mu A}$ propagator in the background field $A_{\mu a}$, and the $\psi_A\phi_A$ propagator in the background field $A_{\mu a}$, are fully expressed as sums over paths, weighted by the path-ordered phase-factor in the representation $(t_a)_{CD}=f_{CaD}$ of G.

We note furthermore that $\frac{1}{4g^2}(F_{\mu\nu a}F_{\mu\nu a}+F_{\mu\nu A}F_{\mu\nu A})$ is equal to the sum of the $A_{\mu a}$'s Yang Mills action density $\frac{1}{4g^2}\bar{F}_{\mu\nu a}\bar{F}_{\mu\nu a}$, plus $\frac{1}{4g^2}$ times the $A_{\mu A}$'s kinetic terms (1.12) in the background field $A_{\mu a}$, plus the manifestly G-gauge-invariant interaction terms:

$$\frac{1}{4g^2}(4(\bar{D}_\mu A_\nu)_A f_{ABC}A_{\mu B}A_{\nu C}+(f_{aBC}f_{aEF}+f_{ABC}f_{AEF})A_{\mu B}A_{\nu C}A_{\mu E}A_{\nu F}) \tag{1.31}$$

1.4 Group-Changing Equations for theVacuum Expectation Values and Correlation Functions for the Group, in terms of those for the Subgroup

We now consider the calculation of the vacuum expectation value, in the Yang Mills theory for the compact Lie algebra H, of a general H-gauge-invariant quantity, formed from the path-ordered phase factors, in various representations of H, that correspond to the paths in some network of paths, where the paths in the network meet at junctions, and the path-ordered phase factors are contracted at the junctions with H-invariant tensors with indices in the appropriate representations of H. We use the following procedure: we *first* functionally integrate over the $A_{\mu A}$, B_A, ψ_A, and ϕ_A fields, in the presence of a general background $A_{\mu a}$ field configuration, which we do *not* functionally integrate over at this stage. We use Landau gauge for the $A_{\mu A}$ and B_A fields, or in other words, we set β equal to 0 in (1.6). Then, as shown above, we may use the manifestly G-gauge-invariant form (1.23) for the $\phi_A\psi_B$ action. There are now *no* terms in the action that contain both one or more of the fields $A_{\mu A}$, B_A, ψ_A, and ϕ_A, and one or more of the fields B_a, ψ_a, and ϕ_a, (thus it does not matter whether or not the B_a, ψ_a, and ϕ_a fields are integrated over at this stage). The exact $A_{\mu A}A_{\nu B}$ and $\psi_A\phi_B$ propagators in the background field $A_{\mu a}$ are expressed as sums over paths, weighted by the G-covariant path-ordered phase factors in the representation $(t_a)_{CD} = f_{CaD}$ of G, by (1.18), (1.20), (1.27), (1.29), and (1.30), while the $A_{\mu A}B_B$ and B_AB_B propagator components are irrelevant, since the B_A field is not involved at all in this vacuum expectation value.

In each H-covariant path-ordered phase factor in the H-invariant quantity whose vacuum expectation value we are calculating, we put $A_{\mu\alpha}(t_\alpha)_{ij} = A_{\mu a}(t_a)_{ij} + A_{\mu A}(t_A)_{ij}$, and expand our H-invariant quantity in powers of $A_{\mu A}$. We then see immediately from (1.1) that between each occurrence of $A_{\mu A}(t_A)_{ij}$ along the path-ordered phase factor in the representation t_α of H, we have a *G-covariant* path-ordered phase-factor in the representation t_a of G.

Furthermore, $(t_A)_{ij}$, considered as a tensor in its three indices A, i, and j, is an invariant tensor of G, where the index A is in the representation f_{AaB} of G, the index i is in the representation t_a of G, and the index j is in the complex conjugate representation $(t_a)^*$ of G. Indeed, the commutation relation $[t_a, t_A] = f_{aAB}t_B$ implies immediately that

for any infinitesimal parameters ϵ_a, $a \in G$, we have:

$$(\delta_{ik} + (\epsilon_a t_a)_{ik})(\delta_{jm} + \epsilon_b((t_b)_{jm})^*)(\delta_{AB} + \epsilon_c f_{AcB})(t_B)_{km} = (t_A)_{ij} \tag{1.32}$$

And similarly, we find from the commutation relation (1.4) that

$$\epsilon_a f_{CaE} f_{EbD} + \epsilon_a f_{DaE} f_{CbE} + \epsilon_a f_{bae} f_{CeD} = 0$$

hence that f_{CbD} is an invariant tensor of G, and by taking, in the general Jacobi identity $f_{\beta\alpha\epsilon} f_{\epsilon\gamma\delta} + f_{\gamma\alpha\epsilon} f_{\beta\epsilon\delta} + f_{\delta\alpha\epsilon} f_{\beta\gamma\epsilon} = 0$ of H, $\alpha = a$, $\beta = B$, $\gamma = C$, and $\delta = D$, and recalling that the structure constants with one upper-case index and two lower-case indices are all equal to zero, we find that $f_{BaE} f_{ECD} + f_{CaE} f_{BED} + f_{DaE} f_{BCE} = 0$, hence that f_{BCD} is an invariant tensor of G.

We now develop the standard perturbation expansion for the functional integral over the $A_{\mu A}$, B_A, ψ_A, and ϕ_A fields, with the pre-exponential factor given by our H-invariant quantity, and in the presence of the general $A_{\mu a}$ "background field" configuration, which we do *not* yet functionally integrate over, and we find immediately from the foregoing that we have a sum over "decorations" of our H-gauge-invariant quantity by new paths and junctions, where all the paths are now in the appropriate representations of G, and contracted at the junctions by the appropriate G-invariant tensors, and all new paths are to be summed over with the appropriate position-space weight according to which of our propagators in the background field $A_{\mu a}$ they represent, and all new junctions are to be integrated, as appropriate, either over all four space dimensions or, if they belong to a path of our H-invariant quantity, along that path, respecting path-ordering along that path with any other new junctions that belong on that path.

Let us denote the vacuum functional integral over *all* our fields, (both upper-case index and lower-case index), by Z_H. We do *not* yet divide by Z_H. Therefore we must also include all vacuum bubbles, in the presence of the general background field configuration $A_{\mu a}$. Each vacuum bubble consists of a G-invariant network of paths and junctions formed from G-covariant path-ordered phase factors and G-invariant tensors just as before, with the paths being summed over and the junction positions integrated over, with the only difference being that there are now *no* fixed paths or junctions, (and *all* paths are now in the representation f_{AaB} of G). And we treat Z_H in exactly the same fashion, first functionally integrating just over the $A_{\mu A}$, B_A, ψ_A, and ϕ_A fields, in the presence of a general background field $A_{\mu a}$, and develop Z_H as a sum over vacuum bubbles in the presence of $A_{\mu a}$.

Then, for each separate term in each of the above expansions, we functionally integrate over the $A_{\mu a}$ field, (and also over the B_a, ψ_a, and ϕ_a fields, if that has not already been done). Furthermore, we divide every term by the vacuum functional integral over the fields $A_{\mu a}$, B_a, ψ_a, and ϕ_a, which we denote by Z_G. Thus all vacuum bubbles due to the *lower*-case index fields propagating as quantum fields in loops are cancelled out, while all the vacuum bubbles due to the *upper*-case index fields propagating as quantum fields in loops, remain.

1.4.1 Emergence of correlation functions involving vacuum bubbles in the subgroup

Let W represent our initial H-invariant quantity. Let $[W]_H$ denote $\frac{1}{Z_G}$ times the functional integral over *all* the fields, with the pre-exponential factor given by W, where the subscript H indicates that we have done the functional integral over *all* the fields, and the square brackets are to remind us that we have divided by Z_G, not by Z_H. And let $\tilde{Z}_H$ denote $\frac{Z_H}{Z_G}$.

We then see that for every term in each of our expansions, and for every position-space configuration of the new paths and junctions that occur in that term, we have the *vacuum expectation value in the Yang Mills theory for* G of the G-invariant quantity that corresponds to that term, and to that configuration of the new paths and junctions of that term.

Now the vacuum expectation value of W *in the Yang Mills theory for* H is given by $\frac{[W]_H}{\tilde{Z}_H}$.

Suppose $[W]_H$ contains a term $\langle W_1 W_2 \rangle_G$, where W_1 is a G-invariant, decorated version of W, and W_2 is a G-invariant vacuum bubble, and the subscript G indicates the vacuum expectation value in the Yang Mills theory for G. Then $[W]_H$ also contains the term $\langle W_1 \rangle_G$, and $\tilde{Z}_H$ includes the term $\langle W_2 \rangle_G$, so that $\frac{1}{\tilde{Z}_H}$, expanded in powers of $(\tilde{Z}_H - 1)$, includes the term $-\langle W_2 \rangle_G$. Thus the total of all the terms in $\frac{[W]_H}{\tilde{Z}_H}$ that contain precisely the two G-invariants W_1 and W_2, is $\langle W_1 W_2 \rangle_G - \langle W_1 \rangle_G \langle W_2 \rangle_G$, which is the *correlation function* of W_1 and W_2 in the Yang Mills theory for G. And similarly, suppose $[W]_H$ contains a term $\langle W_1 W_2 W_3 \rangle_G$, where of W, and W_2 and W_3 are G-invariant vacuum bubbles. Then $[W]_H$ also includes terms $\langle W_1 \rangle_G$, $\langle W_1 W_2 \rangle_G$, and $\langle W_1 W_3 \rangle_G$, and $\tilde{Z}_H$ contains terms $\langle W_2 \rangle_G$, $\langle W_3 \rangle_G$, and $\langle W_2 W_3 \rangle_G$, hence $\frac{1}{\tilde{Z}_H}$ includes terms $+2\langle W_2 \rangle_G \langle W_3 \rangle_G$, (from $+(\tilde{Z}_H - 1)^2$), and $-\langle W_2 W_3 \rangle_G$, (from $-(\tilde{Z}_H - 1)$). Hence

the total of all the terms in $\frac{[W]_H}{\tilde{Z}_H}$ that contain precisely the three G-invariants W_1, W_2, and W_3, is:

$$\langle W_1W_2W_3\rangle_G - \langle W_1W_2\rangle_G\langle W_3\rangle_G - \langle W_1W_3\rangle_G\langle W_2\rangle_G - \langle W_1\rangle_G\langle W_2W_3\rangle_G +$$

$$+2\langle W_1\rangle_G\langle W_2\rangle_G\langle W_3\rangle_G$$

which is the *correlation function* of W_1, W_2, and W_3, in the Yang Mills theory for G.

And in general, if W_1 is a G-invariant decoration of W, and W_2,...,W_n are G-invariant vacuum bubbles, then the total of all the terms in $\frac{[W]_H}{\tilde{Z}_H}$ that contain precisely the n G-invariants W_1, W_2,...,W_n, is equal to the sum, over all *partitions* of the set $\{W_1, W_2, \ldots, W_n\}$, of $(-1)^{m-1}(m-1)!$, where m is the number of parts of the partition, times the product, overthe parts of the partition, of the vacuum expectation value, in the Yang Mills theory for G, of the W_i's in that part of the partition. And this is precisely the correlation function, in the Yang Mills theory for G, of the n G-invariants W_1, W_2,...,W_n. (We note that each partition into m parts comes from a term in $[W]_H$ times a term in $(-1)^{m-1}(\tilde{Z}_H - 1)^{m-1}$.)

Now if W is itself a product of two or more H-invariant quantities, then we may wish to calculate the correlation function of those quantities, in the Yang Mills theory for H. For example, if W is equal to the product of W_1 and W_2, where W_1 and W_2 are H-invariant quantities, then the correlation function of W_1 and W_2, in the Yang Mills theory for H, is equal to $\frac{[W_1W_2]_H}{\tilde{Z}_H} - \frac{[W_1]_H}{\tilde{Z}_H}\frac{[W_2]_H}{\tilde{Z}_H}$. Each of the three vacuum expectation values, in the Yang Mills theory for H, that occurs here, may be expressed in terms of vacuum expectation values, in the Yang Mills theory for G, of G-invariant decorations of W_1W_2, by the results already obtained. Now a decoration of W_1W_2 may connect W_1 and W_2 into a single G-invariant quantity, say W_3, or W_1 and W_2 may be decorated into two separate G-invariant quantities, say W_3 and W_4, and in the latter case, $[W_1]_H$ contains the term $\langle W_3\rangle_G$ and $[W_2]_H$ contains the term $\langle W_4\rangle_G$, so that we obtain the correlation function of W_3 and W_4 in the Yang Mills theory for G. Now suppose, for example, that $[W_1W_2]_H$ contains the term $\langle W_3W_4W_5\rangle_G$, where W_3 is a G-invariant decoration of W_1, W_4 is a G-invariant decoration of W_2, and W_5 is a G-invariant vacuum bubble. Then $[W_1]_H$ contains the terms $\langle W_3\rangle_G$ and $\langle W_3W_5\rangle_G$, $[W_2]_H$ contains the terms $\langle W_4\rangle_G$ and $\langle W_4W_5\rangle_G$, and $\tilde{Z}_H$ contains the term $\langle W_5\rangle_G$, from which we find immediately that the total of all the terms in $\frac{[W_1W_2]_H}{\tilde{Z}_H} - \frac{[W_1]_H}{\tilde{Z}_H}\frac{[W_2]_H}{\tilde{Z}_H}$ that contain precisely the three G-invariants W_3, W_4, and W_5, is the correlation function of W_3, W_4, and W_5, in the Yang Mills theory for G.

And in general, if we calculate the correlation function, in the Yang Mills theory for H, of n separate H-invariant quantities W_1,...,W_n, and use our previous results to express all the vacuum expectation values, in the Yang Mills theory for H, of subsets of $\{W_1, \ldots, W_n\}$, in terms of correlation functions, in the Yang Mills theory for G, of G-invariant decorations of those subsets of $\{W_1, \ldots, W_n\}$, and G-invariant vacuum bubbles, then we find that if P is a partition into m parts of the set $\{W_1, \ldots, W_n\}$, (so that $1 \leq m \leq n$ holds), and $\tilde{W}_1$, ..., $\tilde{W}_m$ are G-invariant *connected* decorations of the parts of P, and $\tilde{W}_{m+1}$, ..., $\tilde{W}_{m+r}$ are G-invariant vacuum bubbles, then the total of all the terms in the correlation function of $\{W_1, \ldots, W_n\}$ in the Yang Mills theory for H, that involve precisely the $(m+r)$ G-invariant quantities $\tilde{W}_1$, ..., $\tilde{W}_{m+r}$, is equal to the correlation function, in the Yang Mills theory for G, of $\{\tilde{W}_1, \ldots, \tilde{W}_{m+r}\}$.

1.4.2 One-loop vacuum bubbles in the subgroup

The one-loop vacuum bubbles require special treatment, because they have no junctions. For the Fadeev-Popov one-loop vacuum bubble we have tr $\ln(\bar{D}^2) = \ln \det(\bar{D}^2)$, while for the $A_{\mu A}$ and B_A one-loop vacuum bubble we have $-\frac{1}{2}$ times the trace of the logarithm of the matrix of G-covariant derivatives that occurs at the left of (1.14) and (1.15), with β set equal to 0.

For tr $\ln(\bar{D}^2)$ we use:

$$\int_0^\infty \frac{ds}{s} \left(e^{s\bar{D}^2} - e^{s\partial^2} \right) = -\ln \left(\bar{D}^2 \left(\frac{1}{\partial^2} \right) \right) \tag{1.33}$$

We then treat $e^{s\bar{D}^2}$ in the same manner as before, with the result being obtained from (1.27) by removing the overall factor of σ, dividing term n by $(n+1)$, setting x equal to y and integrating, and taking the trace on the group indices.

For the $A_{\mu A}$ and B_A one-loop vacuum bubble the calculation is facilitated by use of the identity:

$$\begin{aligned} &\text{tr } \ln(MN) = \ln \det(MN) = \ln(\det M \det N) = \\ &= (\ln \det M) + (\ln \det N) = (\text{tr } \ln M) + (\text{tr } \ln N) \end{aligned} \tag{1.34}$$

which holds for general M and N.

Now let M denote the matrix of G-covariant derivatives that occurs at the left of (1.14) and (1.15), with β set equal to 0, so that the one-loop vacuum bubble for $A\mu A$

and B_A is given by $-\frac{1}{2}\text{tr ln}M$. Let M_0 denote M with the term $-2\bar{F}_{\mu\sigma}$ removed, let g^2G_0 denote the propagator matrix (1.20), and let F denote the matrix $\left(\begin{array}{c|c} 2\bar{F}_{\mu\sigma} & 0 \\ 0 & 0 \end{array}\right)$.

Then

$$\text{tr ln}M = \text{tr ln}(M_0G_0M) = \text{tr ln}(M_0G_0(M_0 - F)) = \text{tr ln}(M_0(1 - G_0F)) =$$

$$= (\text{tr ln}M_0) - \text{tr}(G_0F) - \frac{1}{2}\text{tr}(G_0FG_0F) - \frac{1}{3}\text{tr}(G_0FG_0FG_0F) - \ldots \tag{1.35}$$

To calculate tr lnM_0 we express M_0 as:

$$\left(\begin{array}{c|c} -\bar{D}^2\delta_{\mu\alpha} & 0 \\ 0 & \frac{1}{\theta} \end{array}\right)\left(\begin{array}{c|c} -\frac{1}{\bar{D}^2}\delta_{\alpha\gamma} & 0 \\ 0 & \theta \end{array}\right)\left(\begin{array}{c|c} -\bar{D}^2\delta_{\gamma\sigma} + \bar{D}_\gamma\bar{D}_\sigma & -i\bar{D}_\gamma \\ i\bar{D}_\sigma & 0 \end{array}\right) \tag{1.36}$$

where θ will be chosen for convenience. Now the product of the second and third matrices in (1.36) is:

$$\left(\begin{array}{c|c} \delta_{\alpha\sigma} - \frac{1}{\bar{D}^2}\bar{D}_\alpha\bar{D}_\sigma & i\frac{1}{\bar{D}^2}\bar{D}_\alpha \\ i\theta\bar{D}_\sigma & 0 \end{array}\right) = \left(\begin{array}{c|c} \delta_{\alpha\sigma} & 0 \\ 0 & 1 \end{array}\right) - \left(\begin{array}{c|c} \frac{1}{\bar{D}^2}\bar{D}_\alpha\bar{D}_\sigma & -i\frac{1}{\bar{D}^2}\bar{D}_\alpha \\ -i\theta\bar{D}_\sigma & 1 \end{array}\right) \tag{1.37}$$

We denote the right-hand side of (1.37) by $1 - N$, and we now choose $\theta = -1$ so that N factorizes as:

$$\left(\begin{array}{c|c} \frac{1}{\bar{D}^2}\bar{D}_\alpha\bar{D}_\sigma & -i\frac{1}{barD^2}\bar{D}_\alpha \\ i\bar{D}_\sigma & 1 \end{array}\right) = \left(\begin{array}{c} \frac{1}{\bar{D}^2}\bar{D}_\alpha \\ i \end{array}\right)\left(\bar{D}_\sigma\ ,\ -i\right) \tag{1.38}$$

We then find immediately that for $n \geq 1$, N^n is equal to:

$$\left(\begin{array}{c} \frac{1}{\bar{D}^2}\bar{D}_\alpha \\ i \end{array}\right)(E+1)^{-1}(E+1)^n\left(\bar{D}_\sigma\ ,\ -i\right) \tag{1.39}$$

hence

$$\ln(1-N) = -N - \frac{1}{2}N^2 - \frac{1}{3}N^2 - \ldots =$$

$$= \left(\begin{array}{c} \frac{1}{\bar{D}^2}\bar{D}_\alpha \\ i \end{array}\right)(E+1)^{-1}\left(-(E+1) - \frac{1}{2}(E+1)^2 - \frac{1}{3}(E+1)^3 - \ldots\right)\left(\bar{D}_\sigma\ ,\ -i\right)$$

$$= \left(\begin{array}{c} \frac{1}{\bar{D}^2}\bar{D}_\alpha \\ i \end{array}\right)(E+1)^{-1}(\ln(-E))\left(\bar{D}_\sigma\ ,\ -i\right) \tag{1.40}$$

hence

$$\text{tr ln}(1-N) = \text{tr ln}(-E) \tag{1.41}$$

hence

$$\text{tr } \ln M_0 = 4\text{tr } \ln(\bar{D}^2) + \text{tr } \ln E + \text{constant} \tag{1.42}$$

where the $A_{\mu A}$-independent constant cancels between $[W]_H$ and $\tilde{Z}_H$ for any W, due to exponentiation of the vacuum bubbles.

We call the equations we have derived in this section, which express the vacuum expectation values and correlation functions of H-invariant quantities, in the Yang Mills theory for H, in terms of the vacuum expectation values and correlation functions of G-invariant quantities, in the Yang Mills theory for the Lie subalgebra G of H, the *group-changing equations for G and H*.

In the applications we need to consider path-ordered phase factors having an insertion of $F_{\mu\nu a}(t_a)_{ij}$ at a finite number of points along the path, where t_a is the representation appropriate to that path, but these can all be treated by the principles already given.

1.5 The Group-Changing Equations for $\mathbf{SU}(NM)$ and $(\mathbf{SU}(N))^M$

In this section we apply the group-changing equations to the groups $G = (\text{SU}(N))^M$ and $H = \text{SU}(NM)$, then take the derivative with respect to M at $M = 1$, to obtain a complete and closed set of equations among the coefficients of the $\frac{1}{N}$ expansions of the one-Wilson-loop vacuum expectation value and the multi-Wilson-loop correlation functions in SU(N) Yang Mills theory. In the next section we shall see that the equations for the leading terms in the $\frac{1}{N}$ expansions have a very simple structure, and that their solution will manifestly satisfy the Wilson area law for the one-Wilson-loop vacuum expectation value, and the requirement of massive glueball saturation for the multi-Wilson-loop correlation functions.

1.5.1 Review of the $\frac{1}{N}$ expansion

We begin be recalling some well-known facts about SU(N) Yang Mills theory [32]. For any integers α, β, we define $\theta_{\alpha\beta}$ to be equal to 1 if $\alpha < \beta$ and equal to 0 if $\alpha \geq \beta$. Then as a basis for the fundamental representation of SU(N) we may take, with $1 \leq j \leq N$,

$1 \leq k \leq N$:

$$\begin{aligned}
(t_{1ab})_{jk} &= \frac{i}{\sqrt{2}}(\delta_{aj}\delta_{bk} + \delta_{ak}\delta_{bj}) && (1 \leq a < b \leq N) \\
(t_{2ab})_{jk} &= \frac{1}{\sqrt{2}}(\delta_{aj}\delta_{bk} - \delta_{ak}\delta_{bj}) && (1 \leq a < b \leq N) \\
(t_{3a})_{jk} &= \frac{i}{\sqrt{a(a-1)}}\delta_{jk}(\theta_{ja} - (a-1)\delta_{ja}) && (2 \leq a \leq N) \qquad (1.43)
\end{aligned}$$

where the t_1's, t_2's, and t_3's are repectively generalizations of $\frac{i}{\sqrt{2}}$ times the Pauli matrices σ_1, σ_2, and σ_3.

We let greek indices run over all the $N^2 - 1$ generators of SU(N), that is, over the $\frac{1}{2}N(N-1)$ t_1's, the $\frac{1}{2}N(N-1)$ t_2's, and the $N-1$ t_3's. We then find that the generators satisfy:

$$\mathrm{tr}(t_\alpha t_\beta) = -\delta_{\alpha\beta} \qquad (1.44)$$

which immediately implies, from the commutation relation $[t_\alpha, t_\beta] = f_{\alpha\beta\gamma}t_\gamma$, that:

$$f_{\alpha\beta\gamma} = -\,\mathrm{tr}([t_\alpha, t_\beta]t_\gamma) = -\,\mathrm{tr}(t_\alpha t_\beta t_\gamma - t_\gamma t_\beta t_\alpha) \qquad (1.45)$$

hence that the structure constants are totally antisymmetric.

We now find, with the summation convention temporarily suspended, that:

$$\sum_{1\leq a<b\leq N} (t_{1ab})_{jk}(t_{1ab})_{pq} = -\frac{1}{2}(\delta_{jp}\delta_{kq} + \delta_{jq}\delta_{kp})(\theta_{jk} + \theta_{kj}) \qquad (1.46)$$

$$\sum_{1\leq a<b\leq N} (t_{2ab})_{jk}(t_{2ab})_{pq} = -\frac{1}{2}(\delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp})(\theta_{jk} + \theta_{kj}) \qquad (1.47)$$

$$\begin{aligned}
\sum_{1\leq a<b\leq N} ((t_{1ab})_{jk}(t_{1ab})_{pq} + (t_{2ab})_{jk}(t_{2ab})_{pq}) &= -\delta_{jq}\delta_{kp}(\theta_{jk} + \theta_{kj}) \\
&= -\delta_{jq}\delta_{pk}(1 - \delta_{jk}) \qquad (1.48)
\end{aligned}$$

$$\sum_{2\leq a\leq N} (t_{3a})_{jk}(t_{3a})_{pq} = -\delta_{jk}\delta_{pq} \sum_{2\leq a\leq N} \frac{1}{a}\left(\frac{\theta_{ja}\theta_{pa}}{(a-1)} - \delta_{ja}\theta_{pa} - \delta_{pa}\theta_{ja} + (a-1)\delta_{ja}\delta_{pa}\right)$$

$$= -\delta_{jk}\delta_{pq}\left(\left(\begin{array}{c} \sum \\ \max(j,p)+1 \leq a \\ a \leq N \end{array} \left(\frac{1}{a-1} - \frac{1}{a}\right)\right) - \frac{\theta_{pj}}{j} - \frac{\theta_{jp}}{p} + \left(\frac{j-1}{j}\right)\delta_{jp}\right)$$

$$= -\delta_{jk}\delta_{pq}\delta_{jp} - \delta_{jk}\delta_{pq}\left(\left(\frac{1}{\max(j,p)} - \frac{1}{N}\right) - \frac{1}{\max(j,p)}\right) = -\delta_{jq}\delta_{pk}\delta_{jk} + \frac{1}{N}\delta_{jk}\delta_{pq} \quad (1.49)$$

hence, with the usual summation convention, and recalling our convention that the greek indices run over *all* the generators of SU(N):

$$(t_\alpha)_{jk}(t_\alpha)_{pq} = -\delta_{jq}\delta_{pk} + \frac{1}{N}\delta_{jk}\delta_{pq} \quad (1.50)$$

(We note that $(t_\alpha)_{jk}(t_\alpha)_{pq}$ is an invariant tensor of SU(N), and that once it is known that $\delta_{jq}\delta_{pk}$ and $\delta_{jk}\delta_{pq}$ are the only invariant tensors that occur in the right-hand side of (1.50), then the coefficients may be determined directly from (1.44) and the tracelessness of the t_α's. But for other groups and representations, further linearly independent invariant tensors might occur in the right-hand side of (1.50).)

We replace the coupling constant g of the preceding sections by $\frac{g}{\sqrt{N}}$, so that the Yang Mills action density is now given by:

$$\frac{N}{4g^2}F_{\mu\nu\alpha}F_{\mu\nu\alpha} \quad (1.51)$$

Then, in each Feynman diagram that contributes to a vacuum expectation value or correlation function of Wilson loops, we use the relation (1.45) to express all the structure constants in terms of the t_α's, then use the relation (1.50) and the $\delta_{\alpha\beta}$ colour structure of the propagators, (which has the consequence that the t_α's all occur in pairs with the adjoint representation indices contracted as in the left-hand side of (1.50)), to eliminate all the t_α's in terms of fundamental representation Kronecker deltas. Then for each independent choice of one of the two terms in (1.45) for each structure constant, and one of the two terms in (1.50) for each propagator and for the $\delta_{\alpha\beta}$ in the middle of each four-gluon vertex, we obtain a collection of closed loops of fundamental representation Kronecker deltas, each of which simply gives a factor N. The total N-dependence of each such term is given by a factor N for each vertex *that comes from the action*, a factor $\frac{1}{N}$ for each propagator, a factor $\frac{1}{N}$ for each choice of the second term in the right-hand side of (1.50), and a factor N for each closed loop of fundamental representation Kronecker deltas. (There is *no* factor associated with each vertex where a gluon line ends at a Wilson loop.)

We note that replacing the first term in (1.50) by the second term in (1.50) in any one location, (i.e. in any one propagator or four-gluon vertex), either leaves the total power of N unaltered, or else decreases it by 2. For if the two Kronecker deltas in the first term are in the *same* closed loop of Kronecker deltas, then replacing the first term

of (1.50) by the second term in that location, *increases* the number of closed loops of Kronecker deltas by 1, so that the total power of N is unaltered, while if the two Kronecker deltas in the first term are in two *different* closed loops of Kronecker deltas, then replacing the first term by the second in that location *decreases* the number of closed loops of Kronecker deltas by 1, so that the total power of N decrease by 2. Thus we may determine the maximum possible power of N corresponding to any Feynman diagram by choosing the first term in (1.50) at every location.

Now every Feynman diagram that contributes to the vacuum expectation value of n Wilson loops may be built up from the diagram with no propagators by the successive addition of "new" propagators to lower-order diagrams. Each end of the "new" propagator may be either on a Wilson loop, or at an "old" cubic vertex, (which becomes in quartic vertex in the "new" diagram), or in the "middle" of an "old" propagator, (which results in a new cubic vertex, and an increase by 1 in the number of "old" propagators). Thus each such addition of one "new" propagator results in the difference between the *total* number of propagators, and the total number of vertices *coming from the action*, increasing by 1, thus each addition of one "new" propagator brings a factor $\frac{g^2}{N}$ from the explicit factor $\frac{N}{g^2}$ in (1.51). We assume first that we choose the first term in (1.50) in every propagator and every quartic vertex, so that we just have a sum of terms corresponding to which of the two terms in (1.44) is chosen for each structure constant. Then each end of the "new" propagator breaks into an "old" closed loop of Kronecker deltas, and if both ends of the "new" propagator break into the *same* old loop, then the total number of closed loops *increases* by 1, hence the total power of N is *unaltered*, while if the two ends of the "new" propagator break into *different* old loops, then the total number of closed loops of Kronecker deltas *decreases* by 1, hence the total power of N decreases by 2. Hence we see immediately by induction on the power of g^2 that every term has the power of N of the zeroth order term, multiplied by an integer power ≥ 0 of $\frac{1}{N^2}$, and by the preceding paragraph we see immediately that this remains true when we allow the second term in (1.50) as well. We furthermore see by induction, allowing again just the first term in (1.50), that in any term that contributes at the leading power of N, i.e. at the same power of N as in the zeroth order term, every occurrence of the first term in (1.50), has its two Kronecker deltas in *different* closed loops of Kronecker deltas. For the only way to obtain the leading power of N is to ensure that, as the diagram and the term of interest is built up by successive additions of "new" propagators to lower-order diagrams, the two ends of each "new"

propagator break into the *same* closed loop of Kronecker deltas, so that each addition of a "new" propagator increases the total number of closed loops of Kronecker deltas by 1, and this means firstly that the two Kronecker deltas in the "new" propagator are in two different closed loops of Kronecker deltas, and secondly that no propagator of the "new" diagram that formed the whole or part of a propagator of the "old" diagram, has both its Kronecker deltas in a single closed loop of Kronecker deltas, unless that was the case in the "old" diagram, which by the induction assumption was not so. Hence by the preceding paragraph again, we see that when we again allow both terms in (1.50), *no* term that contributes at the leading power of N contains any occurrence of the second term in (1.50).

Now it is impossible to add a "new" propagator to a diagram with two or more connected components, in such a way that each end of the "new" propagator is in a different connected component of the "old" diagram, (and thus the "new" propagator decreases the number of connected components of the diagram by 1), and retain the same power of N as the "old" diagram, for the fact that the two ends of the "new" propagator are in different connected components of the "old" diagram, means that the two ends of the "new" propagator must break into *different* closed loops of Kronecker deltas of the old diagram. Thus we find that the leading power of N of the diagrams that contribute to the vacuum expectation value of n Wilson loops, and have m or fewer connected components, is:

$$n - 2(n - m) = 2m - n \tag{1.52}$$

And in particular, the leading power of N of the diagrams that contribute to the vacuum expectation value of n Wilson loops and have *one* connected component, or in other words, of the diagrams that contribute to the correlation function of n Wilson loops, is:

$$2 - n \tag{1.53}$$

Now by repeating the same arguments as before, we find that in any term that contributes to the correlation function of n Wilson loops with the largest possible power of N, or in other words, with $(2 - n)$ powers of N, there are *no* occurrences of the second term in (1.50), with one exception: if any propagator, or any $\delta_{\alpha\beta}$ in the middle of a four-gluon vertex, is such that by removing it, the number of connected components of the diagram is increased by 1, then when the first term in (1.50) is taken in that propagator or $\delta_{\alpha\beta}$, the two Kronecker deltas from part of a *single* closed

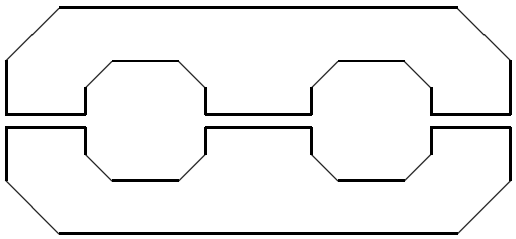
loop of Kronecker deltas, (for any closed loop tha intersects both the "remainder" connected components must have at least two of its Kronecker deltas intersecting both those connected components, and the two Kronecker deltas concerned are the *only* ones that intersect both those connected components), hence when we replace the first term in (1.19) by the second in just that one propagator or $\delta_{\alpha\beta}$, we get a term that exactly cancels the previous term. In fact this argument shows that no Feynman diagram whose number of connected components may be increased by 1 by removing a single propagator or the $\delta_{\alpha\beta}$ from the middle of a single four-gluon vertex, makes any contribution to any correlation function or vacuum expectation value of gauge-invariant quantities, since replacing the first term in (1.50) by the second in that one "key" propagator or $\delta_{\alpha\beta}$, and leaving everything else unchanged, always gives a term that exactly cancels the previous term.

And with this one exception, where, as just noted, replacing the first term in (1.50) by the second term in just one "key" propagator or $\delta_{\alpha\beta}$, results in a term that exactly cancels the previous term, we again find by induction on the power of g^2 that the second term in (1.50) makes *no* contribution to any Feynman diagram term that contributes at the leading power of N, i.e. with $(2-n)$ powers of N, to the correlation function of n Wilson loops.

Now if we consider any Feynman diagram term that contributes at the leading power of N, i.e. with $(2-n)$ powers of N, to the correlation function of n Wilson loops, we may "fill" each closed loop of Kronecker deltas with an oriented topological 2-disk, and we see by induction on the power of g^2 that these oriented topological 2-disks, one for each closed loop of Kronecker deltas, join up to form an oriented 2-sphere with $n$oriented holes in it, where the boundaries of the holes are the n Wilson loops, and the boundaries of the holes are all oriented the same way. In fact, as we build up each diagram from lower-order diagrams by successive addition of "new" propagators, we see that the condition that the two ends of each "new" propagator break into a *single* "old" closed loop of Kroneker deltas, means that each "new" propagator is drawn inside one "window" of the lower-order diagram, or in other words, on the oriented 2-disk that fills the single "old" closed loop of Kronecker deltas into which the "new" propagator breaks. And conversely we see again by induction on the power of g^2, starting from the leading terms, that every connected Feynman diagram that contributes to the correlation function of n Wilson loops and can be drawn on the oriented 2-sphere with n oriented holes of the same orientation, the boundaries of the holes being the Wilson

loops, gives a contribution to that correlation function at the leading power of N, i.e. with $(2-n)$ powers of N, provided that it has no "key" propagator or $\delta_{\alpha\beta}$ in a four-gluon vertex, whose removal increases the number of connected components of the diagram.

For $n \geq 1$ the leading contributions to the correlation function of n Wilson loops have two closed loops of Kronecker deltas and n powers of g^2, and have the general form shown here:



(1.54)

where we have shown the closed loops of Kronecker deltas for the case $n = 3$, and the pairs of parallel lines represent gluon propagators, while the single lines represent Wilson loop segments.

Now by the foregoing, the N-dependence of the correlation function of n Wilson loops in SU(N) Yang Mills theory is given, for all $n \geq 1$, by:

$$N^{2-n}(f_0(W_1,\ldots,W_n,g^2)+\frac{1}{N^2}f_1(W_1,\ldots,W_n,g^2)+\frac{1}{N^4}f_2(W_1,\ldots,W_n,g^2)+\ldots) \quad (1.55)$$

where $W_1,\ldots,W_n$ are the n Wilson loops, and we recall that we define the "correlation function" of *one* Wilson loop to be equal to the one-Wilson-loop vacuum expectation value.

We note that if we denote the correlation function of n Wilson loops by $[W_1 \ldots W_n]$, (where this use of square brackets should not be confused with the square brackets used for a different purpose in Section 2), then the n-Wilson-loop vacuum expectation values may be expressed in terms of the n-Wilson loop correlation functions by:

$$< W_1 >= [W_1]$$

$$< W_1W_2 >= [W_1W_2] + [W_1][W_2]$$

$$< W_1W_2W_3 >= [W_1W_2W_3] + [W_1W_2][W_3] + [W_1W_3][W_2] + [W_1][W_2W_3] + [W_1][W_2][W_3] \quad (1.56)$$

hence (1.55) also defines the $\frac{1}{N^2}$ expansion of the n-Wilson-loop vacuum expectation values for all $n \geq 1$.

1.5.2 The $\frac{1}{N}$ expansion in the presence of quarks

The expansion coefficients $f_r(W_1, \ldots, W_n, g^2)$ in (1.55) contain all the dynamics of gluons and glueballs. We note that if quarks had been present, we would have had to write an expansion in powers of $\frac{1}{N}$ in (1.55), rather than an expansion in powers of $\frac{1}{N^2}$. However all QCD calculation can be done by first calculating the expansion coefficients in (1.55) in pure Yang Mills theory, then using these expansion coefficients to weight the sums over quark and antiquark paths when quarks are included. This is because each Wilson loop can play either the role of a glueball state, or the role of a quark/antiquark path. Thus to calculate the effect of an extra quark/antiquark vacuum bubble on any process, for example, we include an extra Wilson loop in the relevant pure Yang Mills theory correlation function, where that extra Wilson loop follows the quark/antiquark path of that extra vacuum bubble, and sum over the paths followed by that extra Wilson loop with the appropriate kinematic weight, which is obtained from the quark propagator in a general background Yang Mills field by a procedure analogous to that used in Section 2:

$$\frac{1}{\gamma.D + m} = (\gamma.D - m)\frac{1}{D^2 + \frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu} - m^2} = -(\gamma.D - m)\int_0^\infty ds\ e^{s(D^2 + \frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu} - m^2)} \tag{1.57}$$

where $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$. The $\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}$ term in the exponent in the right-hand side of (1.57) requires insertions of $F_{\mu\nu}$ at finite numbers of points along the path-ordered phase factors, just as we found in Section 2 for the $A_{\mu A}A_{\mu A}$ propagator in a general $A_{\mu a}$ "background" field, in the context of the group-changing equations. We note that it is the combined effect of a $\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}$ term on one quark or antiquark line and a $\frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}$ term on another quark or antiquark line that produces the "hyperfine" splitting that separates the π from the ρ and the N from the Δ [33]. The large u and d quark masses (about 350 MeV) used in that calculation come from the Wilson loop factors weighting the paths, in the manner indicated later in this paper.

1.5.3 A basis for $\mathbf{SU}(NM)$ suited to the $(\mathbf{SU}(N))^M$ subgroup

We now apply the group-changing equations to the groups $G = (\mathrm{SU}(N))^M$ and $H = \mathrm{SU}(NM)$, to obtain a complete and closed set of equations among the expansion coefficients $f_r(W_1, \ldots, W_n, g^2)$ in (1.55). When we have obtained these equations we will find that we can differentiate with respect to M and then set $M = 1$, (as may be

expected from the analytic dependence on N of the correlation functions (1.55)), but for now we assume that M is an integer.

We choose the following basis for the generators of SU(NM). The rows and columns of the generators in the fundamental representation of SU(NM) are labelled by a lower-case index that runs from 1 to N, and an upper-case index that runs from 1 to M, and we recall that for any integers α, β, we define $\theta_{\alpha\beta}$ to be equal to 1 if $\alpha < \beta$ and equal to 0 if $\alpha \geq \beta$. The generators are:

$$(t_{1Aab})_{JjKk} = \frac{i}{\sqrt{2}}\delta_{JA}\delta_{KA}(\delta_{aj}\delta_{bk} + \delta_{ak}\delta_{bj}) \qquad (1 \leq A \leq M, \quad 1 \leq a < b \leq N)$$

$$(t_{2Aab})_{JjKk} = \frac{1}{\sqrt{2}}\delta_{JA}\delta_{KA}(\delta_{aj}\delta_{bk} - \delta_{ak}\delta_{bj}) \qquad (1 \leq A \leq M, \quad 1 \leq a < b \leq N)$$

$$(t_{3Aa})_{JjKk} = \frac{i}{\sqrt{a(a-1)}}\delta_{JA}\delta_{KA}\delta_{jk}(\theta_{ja} - (a-1)\delta_{ja}) \qquad (1 \leq A \leq M, \quad 2 \leq a \leq N)$$

$$(t_{4AaBb})_{JjKk} = \tfrac{i}{\sqrt{2}}(\delta_{AJ}\delta_{aj}\delta_{BK}\delta_{bk} + \delta_{AK}\delta_{ak}\delta_{BJ}\delta_{bj}) \qquad (1 \leq A < B \leq M, \quad 1 \leq a \leq N, \quad 1 \leq b \leq N)$$

$$(t_{5AaBb})_{JjKk} = \tfrac{1}{\sqrt{2}}(\delta_{AJ}\delta_{aj}\delta_{BK}\delta_{bk} - \delta_{AK}\delta_{ak}\delta_{BJ}\delta_{bj}) \qquad (1 \leq A < B \leq M, \quad 1 \leq a \leq N, \quad 1 \leq b \leq N)$$

$$(t_{6A})_{JjKk} = \frac{i}{\sqrt{A(A-1)N}}\delta_{JK}\delta_{jk}(\theta_{JA} - (A-1)\delta_{JA}) \qquad (2 \leq A \leq M) \qquad (1.58)$$

We observe that the t_1's, t_2's, t_4's, and t_5's here are simply a re-labelling of the "off-diagonal" generators, i.e. the t_1's and t_2's, of (1.43), when the N of (1.43) is replaced by NM, while the t_3's and t_6's here are related by an orthogonal linear transformation to the "diagonal" generators, i.e. the t_3's, of (1.43), when the N of (1.43) is replaced by NM.

We now use the convention that Greek indices run over *all* the $(N^2M^2 - 1)$ generators of SU(NM), that is, over the $\frac{1}{2}MN(N-1)$ t_1's, the $\frac{1}{2}MN(N-1)$ t_2's, the $M(N-1)$ t_3's, the $\frac{1}{2}M(M-1)N^2$ t_4's, the $\frac{1}{2}M(M-1)N^2$ t_5's, and the $(M-1)$ t_6's. Then we find, just as before, that the t_α's satisfy the equation (1.44), which again immediately implies the result (1.45), hence that the structure constants are totally antisymmetric.

Now for each fixed value of A, $1 \leq A \leq M$, the t_{1Aab}'s, t_{2Aab}'s, and t_{3Aa}'s generate a distinct SU(N) subalgebra of SU(NM), hence the set of all the t_1's, t_2's, and t_3's generates an $(\mathrm{SU}(N))^M$ subalgebra of SU(NM). Hence when we apply the results of

Section 2, the lower-case indices of Section 2 run over all the sets $\{(1Aab) \mid 1 \leq A \leq M, 1 \leq a < b \leq N\}$, $\{(2Aab) \mid 1 \leq A \leq M, 1 \leq a < b \leq N\}$, and $\{(3Aa) \mid 1 \leq A \leq M, 2 \leq a \leq N\}$, and the upper-case indices of Section 2 run over all the sets $\{(4AaBb) \mid 1 \leq A < B \leq M, 1 \leq a \leq N, 1 \leq b \leq N\}$, $\{(5AaBb) \mid 1 \leq A < B \leq M, 1 \leq a \leq N, 1 \leq b \leq N\}$, and $\{(6A) \mid 2 \leq A \leq M\}$.

We now find, with the summation convention temporarily suspended, that for each value of A, $1 \leq A \leq M$, we have:

$$\left\{ \sum_{1 \leq a < b \leq N} ((t_{1Aab})_{JjKk}(t_{1Aab})_{PpQq} + (t_{2Aab})_{JjKk}(t_{2Aab})_{PpQq}) + \right.$$

$$\left. + \sum_{2 \leq A \leq N} (t_{3Aa})_{JjKk}(t_{3Aa})_{PpQq} \right\} = \delta_{JA}\delta_{KA}\delta_{PA}\delta_{QA}\left(-\delta_{jq}\delta_{kp} + \frac{1}{N}\delta_{jk}\delta_{pq}\right) \tag{1.59}$$

(which is the relation (1.50) for the number A SU(N) subgroup), and that we also have:

$$\sum_{\substack{1 \leq A < B \leq M \\ 1 \leq a \leq N \\ 1 \leq b \leq N}} \left((t_{4AaBb})_{JjKk}\,(t_{4AaBb})_{PpQq} + (t_{5AaBb})_{JjKk}\,(t_{5AaBb})_{PpQq}\right) =$$

$$= -\delta_{JQ}\delta_{jq}\delta_{PK}\delta_{pk}\,(1 - \delta_{JK}) \tag{1.60}$$

and also:

$$\sum_{2 \leq A \leq M} (t_{6A})_{JjKk}\,(t_{6A})_{PpQq} = \frac{1}{N}\delta_{JK}\delta_{jk}\delta_{PQ}\delta_{pq}\left(-\delta_{JQ} + \frac{1}{M}\right) =$$

$$= -\frac{1}{N}\delta_{JQ}\delta_{PK}\delta_{JK}\delta_{jk}\delta_{pq} + \frac{1}{NM}\delta_{JK}\delta_{jk}\delta_{PQ}\delta_{pq} \tag{1.61}$$

hence, with the summation convention restored, and recalling that greek indices now run over *all* the generators of SU(NM), we find that:

$$(t_\alpha)_{JjKk}\,(t_\alpha)_{PpQq} = -\delta_{JQ}\delta_{jq}\delta_{KP}\delta_{kp} + \frac{1}{NM}\delta_{JK}\delta_{jk}\delta PQ\delta_{pq} \tag{1.62}$$

which is the analogue of (1.50) for SU(NM) in the present basis.

Chapter 2

Reduction of the Group-Changing Equations for $\mathbf{SU}(NM)$ and $(\mathbf{SU}(N))^M$, to Equations Expressing the Vacuum Expectation Values and Correlation Functions for $\mathbf{SU}(NM)$, in terms of those for $\mathbf{SU}(N)$

Now let $W_1, \ldots, W_n$ be n Wilson loops in the fundamental representation (1.58) of $\mathrm{SU}(NM)$. Then by Section 1.4, the group-changing equations for $(\mathrm{SU}(N))^M$ and $\mathrm{SU}(NM)$ express the correlation function $[W_1 \ldots W_n]_{\mathrm{SU}(NM)}$, in the Yang Mills theory for $\mathrm{SU}(NM)$, in terms of sums of correlation functions $\left[\tilde{W}_1 \ldots \tilde{W}_s\right]_{(\mathrm{SU}(N))^M}$ of connected, $(\mathrm{SU}(N))^M$-gauge-invariant quantities $\tilde{W}_1, \ldots, \tilde{W}_s$, in the Yang Mills theory for $(\mathrm{SU}(N))^M$. We note that $\tilde{W}_1, \ldots, \tilde{W}_s$ are *not* in general Wilson loops, but rather are more general $(\mathrm{SU}(N))^M$-gauge-invariant quantities obtained from the $\mathrm{SU}(NM)$ Wilson loops $W_1 \ldots W_n$ as follows.

We "decorate" the closed paths defining the Wilson loops $W_1 \ldots W_n$ by the addition of new paths and junctions, such that the n separate closed paths defining $W_1 \ldots W_n$

are replaced by m connected "networks" $\tilde{W}_1, \ldots, \tilde{W}_m$ of paths and junctions, where $1 \leq m \leq n$, and for each $1 \leq i \leq m$, the network defining $\tilde{W}_i$ contains at least one of the n original closed paths, and we also include $r \geq 0$ "vacuum bubbles" $\tilde{W}_{m+1}, \ldots, \tilde{W}_{m+r}$, each of which is a connected network of new paths and junctions that does *not* contain any of the n original closed paths.

Each $\tilde{W}_i$ is an $(\mathrm{SU}(N))^M$-gauge-invariant quantity formed from $(\mathrm{SU}(N))^M$- covariant path-ordered phase factors and $(\mathrm{SU}(N))^M$-invariant tensors as follows. Each path that comes from the whole or part of one of the n original closed paths, is in the representation of $(\mathrm{SU}(N))^M$ obtained by restricting the representation (1.58) of $\mathrm{SU}(NM)$ to the subgroup $(\mathrm{SU}(N))^M$, i.e. to the t_1's, t_2's, and t_3's. And each *new* path is in the representation of $(\mathrm{SU}(N))^M$ which in the notation of Section 2 is given by:

$$(t_a)_{AB} = f_{AaB} = -\mathrm{tr}\,(t_A t_a t_B - t_B t_a t_A) \tag{2.1}$$

where the t_α's in the right-hand side of (2.1) are given by (1.58), and a in (2.1) runs over the sets $\{(1Eef)|1 \leq E \leq M, 1 \leq e < f \leq N\}$, $\{(2Eef)|1 \leq E \leq M, 1 \leq e < f \leq N\}$, and $\{(3Ee)|1 \leq E \leq M, 2 \leq e \leq N\}$, (i.e. over the generators of $(\mathrm{SU}(N))^M$), and A and B in (2.1) each runs over all the sets $\{(4EeFf)|1 \leq E < F \leq M, 1 \leq e \leq N, 1 \leq f \leq N\}$, $\{(5EeFf)|1 \leq E < F \leq M, 1 \leq e \leq N, 1 \leq f \leq N\}$, and $\{(6E)|2 \leq E \leq M\}$.

An at each junction where a *new* path ends at one of the n original closed paths, we have the $(\mathrm{SU}(N))^M$-invariant tensor obtained by restricting the fundamental representation (1.58) of $\mathrm{SU}(NM)$) to the t_4's, t_5's, and t_6's, (which gives an $(\mathrm{SU}(N))^M$-invariant tensor by equation (1.32)), and at each junction at which *three* new paths end, we have the $(\mathrm{SU}(N))^M$-invariant tensor:

$$f_{ABC} = -\mathrm{tr}\,(t_A t_B t_C - t_C t_B t_A) \tag{2.2}$$

where the t_α's in (2.2) are given by (1.58), and A, B, and C in (2.2) each runs over the same domain as A and B in (2.1), and at each junction at which *four* new paths end we have either the $(\mathrm{SU}(N))^M$-invariant tensor $f_{AaB}f_{CaD}$, where f_{AaB} is given by (2.1), the sum on a runs over the generators of $(\mathrm{SU}(N))^M$, and A, B, C, and D each runs over the same domain as A and B in (2.1), or else the $(\mathrm{SU}(N))^M$-invariant tensor $f_{ABC}f_{DBE}$, where f_{ABC} is given by (2.1), the sum on B runs over the same domain as A and B in (2.2), and A, C, D, and E each runs over the same domain as A and B in (2.1), and no junction-types are permitted in the networks $\tilde{W}_i$, $1 \leq i \leq (m+r)$, other than those just described.

Now, as just described, the path-ordered phase factor for each *new* path has at each end an index that runs over all the sets $\{(4EeFf)|1 \leq E < F \leq M, 1 \leq e \leq N, 1 \leq f \leq N\}$, $\{(5EeFf)|1 \leq E < F \leq M, 1 \leq e \leq N, 1 \leq f \leq N\}$, and $\{(6E)|2 \leq E \leq M\}$. We observe that f_{AaB} of (2.1) *vanishes* whenever A or B or both is a member of $\{(6E)|2 \leq E \leq M\}$, which is a consequence of the fact that, for each separate SU(N) subgroup, each t_6 is a multiple of the unit matrix for that SU(N) subgroup, so that each t_6 commutes with every generator of the $(\mathrm{SU}(N))^M$ subgroup of SU(NM). (Hence each $A_{\mu(6E)}$ field interacts with *none* of the gauge fields in the $(\mathrm{SU}(N))^M$ subgroup.) It follows immmediately from this that, firstly, the path-ordered phase factor for a new path has *no* matrix elements between any $(6E)$ at one end and any $(4GgHh)$ or $(5GgHh)$ at its other end, and, secondly, the matrix elelments of the path-ordered phase factor for a new path between any $(6E)$ at one end and any $(6F)$ at the other end, are simply given by δ_{EF}. Thus we may treat the t_6's completely separately from the t_4's and t_5's.

We now use (2.1) and (2.2) to express all the $(\mathrm{SU}(N))^M$-invariant tensors that occur at junctions at which three or four new paths end, in terms of traces of the matrices of the SU(NM) fundamental representation (1.58). This has the consequence that, since the $(\mathrm{SU}(N))^M$-invariant tensor at a junction where a new path ends at one of the n original closed paths, is also a t_4, t_5, or t_6, the path-ordered phase factor W_{AB} for each new path is contracted with a t_A and a t_B in the form $W_{AB}\,(t_A)_{JjKk}\,(t_B)_{PpQq}$, where A and B are each summed over the sets $\{(4EeFf)|1 \leq E < F \leq M, 1 \leq e \leq N, 1 \leq f \leq N\}$, $\{(5EeFf)|1 \leq E < F \leq M, 1 \leq e \leq N, 1 \leq f \leq N\}$, and $\{(6E)|2 \leq E \leq M\}$. Now by the preceding paragraph, W_{AB} vanishes if one of A and B is a $(6E)$ and the other is a $(4GgHh)$ or a $(5GgHh)$, and $W_{(6E)(6F)}$ is equal to δ_{EF}, so the total contribution to $W_{AB}\,(t_A)_{JjKk}\,(t_B)_{PpQq}$, of terms involving one or more t_6's, is $\sum_{2 \leq E \leq M} (t_{6E})_{JjKk}\,(t_{6E})_{PpQq}$, which by (1.61) is equal to $\frac{1}{N}\delta_{JK}\delta_{jk}\delta_{PQ}\delta_{pq}\left(-\delta_{JQ} + \frac{1}{M}\right)$.

2.1 Reduction of the new Wilson lines that involve $(\mathrm{SU}(N))^M$ gauge fields

We now change our convention for greek indices again, and define greek indices to run just over the generators of SU(N), as given by (1.43). Then we note that the generators

$(t_{1Aab})_{JjKk}$, $(t_{2Aab})_{JjKk}$, and $(t_{3Aa})_{JjKk}$ of (1.58) can all be expressed in the form:

$$(t_{A\alpha})_{JjKk} = \delta_{JA}\delta_{KA}\,(t_\alpha)_{jk} = \delta_{JA}\delta_{JK}\,(t_\alpha)_{jk} \tag{2.3}$$

where α runs over the sets $\{(1ab)|1 \leq a < b \leq N\}$, $\{(2ab)|1 \leq a < b \leq N\}$, and $\{(3a)|2 \leq a \leq N\}$, and the SU(N) generators $(t_\alpha)_{jk}$ are defined in (1.43). We note that in (2.3), A, J, and K satisfy $1 \leq A \leq M$, $1 \leq J \leq M$, and $1 \leq K \leq M$, and that the summation convention does *not* apply in the right-hand side.

Then the group index and $(\mathrm{SU}(N))^M$-gauge-field structure of the term in $W_{AB}\,(t_A)_{JjKk}\,(t_B)_{PpQq}$ that is of degree u in the $(\mathrm{SU}(N))^M$-gauge-fields may be expressed in the form, for $u \geq 1$:

$$(-1)^u\Big\{A^{(x(s_1))}_{\mu_1(E_1\alpha_1)}A^{(x(s_2))}_{\mu_2(E_2\alpha_2)}\ldots A^{(x(s_u))}_{\mu_u(E_u\alpha_u)}\,(t_A)_{JjKk}\,\mathrm{tr}\,(t_At_{E_1\alpha_1}t_{C_1} - t_{C_1}t_{E_1\alpha_1}t_A)\times$$

$$\times\,\mathrm{tr}\,(t_{C_1}t_{E_2\alpha_2}t_{C_2} - t_{C_2}t_{E_2\alpha_2}t_{C_1})\ldots\mathrm{tr}\left(t_{C_{u-1}}t_{E_u\alpha_u}t_B - t_Bt_{E_u\alpha_u}t_{C_{u-1}}\right)(t_B)_{PpQq}\Big\} \tag{2.4}$$

where (2.1) has been used, and the E_i's, $1 \leq i \leq u$, are to be summed from 1 to M, the α_i's, $1 \leq i \leq u$, are to be summed over the generators of SU(N), (as specified after (2.3)), and the A, B, and the C_i's, $1 \leq i \leq (u-1)$, are to be summed over the sets $\{(4GgHh)|1 \leq G < h \leq M, 1 \leq g \leq N, 1 \leq h \leq N\}$, and $\{(5GgHh)|1 \leq G < h \leq M, 1 \leq g \leq N, 1 \leq h \leq N\}$. (We note that, due to the assumption $u \geq 1$, it immediately follows from the preceding paragraphs that no terms involving any t_6's make any contribution.)

We now apply (2.3) to the $t_{E_i\alpha_i}$'s in (2.4), then use the result (1.60) to perform the sums over A, B, and the C_i's, $1 \leq i \leq (u-1)$, to obtain:

$$(-1)^u\Big\{A^{(x(s_1))}_{\mu_1(E_1\alpha_1)}A^{(x(s_2))}_{\mu_2(E_2\alpha_2)}\ldots A^{(x(s_u))}_{\mu_u(E_u\alpha_u)}\,(t_A)_{JjKk} \quad \times$$

$$\times\Big(\,(t_A)_{R_1r_1S_1s_1}\,(t_{C_1})_{V_1v_1W_1w_1}\left((t_{E_1\alpha_1})_{S_1s_1V_1v_1}\,\delta_{W_1R_1}\delta_{w_1r_1} - \delta_{S_1V_1}\delta_{s_1v_1}\,(t_{E_1\alpha_1})_{W_1w_1R_1r_1}\right)\Big)\times$$

$$\times\Big(\,(t_{C_1})_{R_2r_2S_2s_2}\,(t_{C_2})_{V_2v_2W_2w_2}\left((t_{E_2\alpha_2})_{S_2s_2V_2v_2}\,\delta_{W_2R_2}\delta_{w_2r_2} - \delta_{S_2V_2}\delta_{s_2v_2}\,(t_{E_2\alpha_2})_{W_2w_2R_2r_2}\right)\Big)\times$$

$$\times \quad \ldots \quad \times$$

$$\times\Big(\Big(t_{C_{u-1}}\Big)_{R_ur_uS_us_u}(t_B)_{V_uv_uW_uw_u}\Big((t_{E_u\alpha_u})_{S_us_uV_uv_u}\,\delta_{W_uR_u}\delta_{w_ur_u} - \delta_{S_uV_u}\delta_{s_uv_u}(t_{E_u\alpha_u})_{W_uw_uR_ur_u}\Big)\Big)\times$$

$$\times \quad (t_B)_{PpQq}\Big\} \quad =$$

$$= \quad (-1)^u \Bigg\{ A^{(x(s_1))}_{\mu_1(E_1\alpha_1)} A^{(x(s_2))}_{\mu_2(E_2\alpha_2)} \dots A^{(x(s_u))}_{\mu_u(E_u\alpha_u)} \left(t_A\right)_{JjKk} \quad \times$$

$$\times \Big(\left(t_A\right)_{R_1 r_1 S_1 s_1} \left(t_{C_1}\right)_{V_1 v_1 W_1 w_1} \delta_{S_1 V_1} \delta_{W_1 R_1} \left(\delta_{E_1 S_1} \left(t_{\alpha_1}\right)_{s_1 v_1} \delta_{w_1 r_1} - \delta_{s_1 v_1} \delta_{E_1 W_1} \left(t_{\alpha_1}\right)_{w_1 r_1} \right) \Big) \times$$

$$\times \Big(\left(t_{C_1}\right)_{R_2 r_2 S_2 s_2} \left(t_{C_2}\right)_{V_2 v_2 W_2 w_2} \delta_{S_2 V_2} \delta_{W_2 R_2} \left(\delta_{E_2 S_2} \left(t_{\alpha_2}\right)_{s_2 v_2} \delta_{w_2 r_2} - \delta_{s_2 v_2} \delta_{E_2 W_2} \left(t_{\alpha_2}\right)_{w_2 r_2} \right) \Big) \times$$

$$\times \quad \dots \quad \times$$

$$\times \Big(\left(t_{C_{u-1}}\right)_{R_u r_u S_u s_u} \left(t_B\right)_{V_u v_u W_u w_u} \delta_{S_u V_u} \delta_{W_u R_u} \left(\delta_{E_u S_u} \left(t_{\alpha_u}\right)_{s_u v_u} \delta_{w_u r_u} - \delta_{s_u v_u} \delta_{E_u W_u} \left(t_{\alpha_u}\right)_{w_u r_u} \right) \Big) \times$$

$$\times \quad \left(t_B\right)_{PpQq} \Bigg\} \quad =$$

$$= \quad - \Bigg\{ A^{(x(s_1))}_{\mu_1(E_1\alpha_1)} A^{(x(s_2))}_{\mu_2(E_2\alpha_2)} \dots A^{(x(s_u))}_{\mu_u(E_u\alpha_u)} \delta_{JS_1} \delta_{js_1} \delta_{R_1 K} \delta_{r_1 k} \left(1 - \delta_{JK}\right) \quad \times$$

$$\times \delta_{V_1 S_2} \delta_{v_1 s_2} \delta_{R_2 W_1} \delta_{r_2 w_1} \left(1 - \delta_{V_1 W_1}\right) \dots \delta_{V_{u-1} S_u} \delta_{v_{u-1} s_u} \delta_{R_u W_{u-1}} \delta_{r_u w_{u-1}} \left(1 - \delta_{V_{u-1} W_{u-1}}\right) \times$$

$$\times \delta_{V_u Q} \delta_{v_u q} \delta_{P W_u} \delta_{p w_u} \left(1 - \delta_{V_u W_u}\right) \delta_{S_1 V_1} \delta_{W_1 R_1} \left(\delta_{E_1 S_1} \left(t_{\alpha_1}\right)_{s_1 v_1} \delta_{w_1 r_1} - \delta_{s_1 v_1} \delta_{E_1 W_1} \left(t_{\alpha_1}\right)_{w_1 r_1} \right) \times$$

$$\times \delta_{S_2 V_2} \delta_{W_2 R_2} \left(\delta_{E_2 S_2} \left(t_{\alpha_2}\right)_{s_2 v_2} \delta_{w_2 r_2} - \delta_{s_2 v_2} \delta_{E_2 W_2} \left(t_{\alpha_2}\right)_{w_2 r_2} \right) \quad \times \quad \dots \quad \times$$

$$\times \delta_{S_u V_u} \delta_{W_u R_u} \left(\delta_{E_u S_u} \left(t_{\alpha_u}\right)_{s_u v_u} \delta_{w_u r_u} - \delta_{s_u v_u} \delta_{E_u W_u} \left(t_{\alpha_u}\right)_{w_u r_u} \right) \Bigg\} \quad =$$

$$= \quad - \Bigg\{ A^{(x(s_1))}_{\mu_1(E_1\alpha_1)} A^{(x(s_2))}_{\mu_2(E_2\alpha_2)} \dots A^{(x(s_u))}_{\mu_u(E_u\alpha_u)} \delta_{JQ} \delta_{PK} \left(1 - \delta_{JK}\right) \quad \times$$

$$\times \left(\delta_{E_1 J} \left(t_{\alpha_1}\right)_{j v_1} \delta_{w_1 k} - \delta_{j v_1} \delta_{E_1 K} \left(t_{\alpha_1}\right)_{w_1 k} \right) \left(\delta_{E_2 J} \left(t_{\alpha_2}\right)_{v_1 v_2} \delta_{w_2 w_1} - \delta_{v_1 v_2} \delta_{E_2 K} \left(t_{\alpha_2}\right)_{w_2 w_1} \right) \times$$

$$\times \quad \dots \quad \times \quad \left(\delta_{E_u J} \left(t_{\alpha_u}\right)_{v_{u-1} q} \delta_{p w_{u-1}} - \delta_{v_{u-1} q} \delta_{E_u K} \left(t_{\alpha_u}\right)_{p w_{u-1}} \right) \Bigg\} \quad =$$

$$= \quad -\delta_{JQ} \delta_{PK} \left(1 - \delta_{JK}\right) \left(A^{(x(s_1))}_{\mu_1(J\alpha_1)} \left(t_{\alpha_1}\right)_{j v_1} \delta_{w_1 k} - \delta_{j v_1} A^{(x(s_1))}_{\mu_1(K\alpha_1)} \left(t_{\alpha_1}\right)_{w_1 k} \right) \times$$

$$\times \left(A^{(x(s_2))}_{\mu_2(J\alpha_2)} \left(t_{\alpha_2}\right)_{v_1 v_2} \delta_{w_2 w_1} - \delta_{v_1 v_2} A^{(x(s_2))}_{\mu_2(K\alpha_2)} \left(t_{\alpha_2}\right)_{w_2 w_1} \right) \times$$

$$\times \quad \dots \quad \times \quad \left(A^{(x(s_u))}_{\mu_u(J\alpha_u)} \left(t_{\alpha_u}\right)_{v_{u-1} q} \delta_{p w_{u-1}} - \delta_{v_{u-1} q} A^{(x(s_u))}_{\mu_u(K\alpha_u)} \left(t_{\alpha_u}\right)_{p w_{u-1}} \right) \tag{2.5}$$

where the E_i's, $1 \leq i \leq u$, the α_i's, $1 \leq i \leq u$, A, B, and the C_i's, $1 \leq i \leq (u-1)$, are to be summed over the same domains as in (2.4), the R_i's, S_i's, V_i's, and W_i's, $1 \leq i \leq u$, are to be summed from 1 to M, and the r_i's, s_i's, v_i's, and w_i's, $1 \leq i \leq u$, are to be summed from 1 to N. (We note that after the application of (1.60), the sums on the S_i's and V_i's collapsed to

$$J = S_1 = V_1 = S_2 = V_2 = \dots = S_u = V_u (= Q)$$

and the sums on the R_i's and W_i's collapsed to

$$K = R_1 = W_1 = R_2 = W_2 = \ldots = R_u = W_u (= P)$$

and that the summation convention is *not* to be applied to J and K in the right-hand side of (2.5).)

We now substitute the result (2.5) for each $u \geq 1$ into the definition (1.1) of the path-ordered phase factor in the representation of $(\mathrm{SU}(N))^M$ given by (2.1), where the indices in (2.1) run over the sets specified immediately after (2.1), (and n in (1.1) is re-written as u), and sum over u from 1 to ∞, and also add the $u = 0$ term as given by (1.60), to conclude that if $x(x)$, $0 \leq s \leq 1$, is any new path, and $W((\mathrm{SU}(N))^M, x(s))_{AB}$ denotes its path-ordered phase factor as specified above, (so that A and B each run over the sets $\{(4EeFf) | 1 \leq E \leq F \leq M, 1 \leq e \leq N, 1 \leq f \leq N\}$, $\{(5EeFf) | 1 \leq E \leq F \leq M, 1 \leq e \leq N, 1 \leq f \leq N\}$, and $\{(6E) | 2 \leq E \leq M\}$), then the total contribution to $W((\mathrm{SU}(N))^M, x(s))_{AB} \, (t_A)_{JjKk} \, (t_B)_{PpQq}$ from terms that involve no t_6's, is simply equal to $-\delta_{JQ} \delta_{PK} \, (1 - \delta_{JK})$ times the *product* of the path-ordered phase factor $W\left(A_{\mu(J\alpha)}, x(s)\right)_{jq}$, in the *fundamental* representation of $\mathrm{SU}(N)$, *for the gauge fields of the number* J $\mathrm{SU}(N)$ *subgroup*, and the path-ordered phase factor $W\left(A_{\mu(K\alpha)}, x(1-s)\right)_{pk}$, in the *fundamental* representation of $\mathrm{SU}(N)$, *for the gauge fields of the number* K $\mathrm{SU}(N)$ *subgroup*, where $x(1-s)$ denotes the given path $x(s)$, traversed in the opposite direction. Indeed, when we write out the product of these two path-ordered phase factors, with each being given by equation (1.1), we may collect together, for each $u \geq 1$, all the terms of *total* degree u in the gauge fields, (where the general such term has r $A_{\mu(J\alpha)}$'s, for some $0 \leq r \leq u$, and $(u - r)$ $A_{\mu(K\alpha)}$'s). Then in each such term we replace the integration variables s_i coming from the second path-ordered phase factor by new integration variables $\tilde{s}_i = (1 - s_i)$, then break up the product of the path-ordered integrals coming from the two separate phase factors into a sum of *totally* path-ordered terms, (i.e. such that the set of all the integration variables s_i coming from the first phase factor, and all the integration variables $\tilde{s}_i$ coming from the second phase factor, is totally ordered), and relabel the integration variables in each such term in accordance with the total path ordering in that term. We then see that we have a sum over the 2^u independent choices of specifying, independently for each gauge field along the path, whether that gauge field is an $A_{\mu(J\alpha)}$ or an $A_{\mu(K\alpha)}$, which is precisely what wee have in the right-hand side of (2.5), when we note that

each $A_{\mu(K\alpha)}$ also brings a minus sign, from

$$\frac{dx_{\mu_i}(1-s_i)}{ds_i} = -\frac{dx_{\mu_i}(\tilde{s}_i)}{d\tilde{s}_i}$$

2.1.1 The SU(N) Wilson lines, in a new $(\mathrm{SU}(N))^M$ Wilson line, belong to *different* SU(N)'s

Hence since, as noted above, the total contribution to

$$W((\mathrm{SU}(N))^M, x(s))_{AB}\,(t_A)_{JjKk}\,(t_B)_{PpQq}$$

from terms that involve one or more t_6's, is simply equal to $\frac{1}{N}\delta_{JK}\delta_{jk}\delta_{PQ}\delta_{pq}\left(-\delta_{JQ}+\frac{1}{M}\right)$, we finally find that, for any new path $x(s)$:

$$W((\mathrm{SU}(N))^M, x(s))_{AB}\,(t_A)_{JjKk}\,(t_B)_{PpQq} =$$

$$= \Big\{ -\delta_{JQ}\delta_{PK}\,(1-\delta_{JK})\,W(A_{\mu(J\alpha)}, x(s))_{jq} W(A_{\mu(K\alpha)}, x(1-s))_{pk} + \\ + \frac{1}{N}\delta_{JK}\delta_{jk}\delta_{PQ}\delta_{pq}\left(-\delta_{JQ}+\frac{1}{M}\right)\Big\} \tag{2.6}$$

where in the left-hand side here, A and B are each summed over the sets $\{(4EeFf)|1 \leq E < F \leq M, 1 \leq e \leq N, 1 \leq f \leq N\}$, $\{(5EeFf)|1 \leq E < F \leq M, 1 \leq e \leq N, 1 \leq f \leq N\}$, and $\{(6E)|2 \leq E \leq M\}$, and the meanings of all the terms in the right-hand side have been explained above.

(We note that in equation (2.6), as in equation (1.1), s is not an argument of either side of the equation - we wrote $x(s)$ simply to display the fact that the argument x of W is a path, not a point.)

We shall refer to the first term in the right-hand side of (2.6), which comes from all the terms in the left-hand side which involve *no* t_6's, as the *45-term*, and the second term in the right-hand side of (2.6), which comes from all the terms in the left-hand side which involve one or more t_6's, as the *6-term*.

We shall also refer to the $A_{\mu(4EeFf)}$ fields, the $A_{\mu(5EeFf)}$ fields, and the $A_{\mu(6E)}$ fields, respectively, and also the corresponding Fadeev-Popov fields, as the 4-fields, the 5-fields, and the 6-fields, respectively.

We note that the crucial $(1-\delta_{JK})$ factor in the 45-term, which results in the 45-term vanishing whenever $J = K$, comes from the fact that the 4-fields and 5-fields interact with two *distinct* SU(N) subgroups.

And we note that the summation convention does *not* apply to the right-hand side of equation (2.6).

2.2 Reduction of the $(\mathrm{SU}(N))^M$ quantities, to sums of products of Wilson loops in the $\mathrm{SU}(N)$'s

We now return to the discussion, which we began after equation (1.62), of the particular group-changing equaton for $\mathrm{SU}(N))^M$ and $\mathrm{SU}(NM)$, that applies to the correlation function $[W_1 \dots W_n]_{\mathrm{SU}(NM)}$, in the Yang Mills theory for $\mathrm{SU}(NM)$, of n Wilson loops $W_1, \dots, W_n$, each in the fundamental representation (1.58) of $\mathrm{SU}(NM)$, and we consider a term $[\tilde{W}_1 \dots \tilde{W}_s]_{\mathrm{SU}(N))^M}$ in the right-hand side of this equation, where each $\tilde{W}_i$ is a connected $\mathrm{SU}(N))^M$-gauge-invariant quantity formed, as described in the discussion following equation (1.62), from $(\mathrm{SU}(N))^M$-covariant path-ordered phase factors corresponding to paths that form parts or wholes of the n original closed paths, (each taken in the representation (2.3) of $(\mathrm{SU}(N))^M$), form $(\mathrm{SU}(N))^M$-covariant path-ordered phase factors corresponding to "new" paths, (each takenin the representation (2.1) of $(\mathrm{SU}(N))^M$), and from the particular $\mathrm{SU}(N))^M$-invariant tensors, at the junctions of the paths, specified in the discussion accompanying equation (2.2).

We note that each vertex involving new paths *only*, comes either from a term in (1.31) or from a term in (1.23). We have to sum over assignments of which of the new paths are gauge-field paths and which are Fadeev-Popov paths, subject to the requirements that there are either 0 or 2 Fadeev-Popov path-ends at any vertex that involves new paths ony, and no Fadeev-Popov path ends on any original path. We consider some specific such assignment.

We now, as described in the paragraphs preceding equation (2.3), use equations (2.1) and (2.2) to express all the $\mathrm{SU}(N))^M$-invariant tensors that occur at junctions at which three or four new paths end, in terms of traces of the matrices of the $\mathrm{SU}(NM)$ fundamental representation (1.58), which, as noted before, has the consequence that every new path has its path-ordered phase factor $W((\mathrm{SU}(N))^M, x(s))_{AB}$ contracted with a t_A and a t_B in the form $W((\mathrm{SU}(N))^M, x(s))_{AB} \, (t_A)_{JjKk} \, (t_B)_{PpQq}$, where A and B are each summed over the sets $\{(4EeFf) | 1 \leq E < F \leq M, 1 \leq e \leq N, 1 \leq f \leq N\}$, $\{(5EeFf) | 1 \leq E < F \leq M, 1 \leq e \leq N, 1 \leq f \leq N\}$, and $\{(6E) | 2 \leq E \leq M\}$, which is exactly what we have in the left-hand side of equation (2.6). We therefore apply (2.6) to every new path.

To see what we get, let us first take the *first* term in the right-hand side of (2.6), i.e. the 45-term, in every new path. We then see immediately that we get a finite number fo terms, each corresponding to an independent choice, at each vertex at which three

new paths meet, of one of the two terms in the right-hand side of (2.2), and also to an independent choice, at each vertex with four gauge-field path-ends, of either the $f_{aBC}f_{aEF}$ term or the $f_{ABC}f_{AEF}$ term in the "quartic vertex" term in (1.31), (while at a vertex with two gauge-field path-ends and two Fadeev-Popov path-ends, we must take the $f_{AaB}f_{CaD}$ quartic vertex term in (1.23), and alsto to independent choices for each of the structure constants occurring in the chosen terms in the quartic vertices, of either of the two terms in (2.1) or in (2.2). Each such choice corresponds to a different routeing, through the relevant vertex, of the index connections of the two separate path-ordered phase factors, each in the fundamental representation of $SUN(N)$, but involving the fields of two *different* SU(N) subgroups of $(\mathrm{SU}(N))^M$, (and traversing their path inn opposite directions), that are multiplied together in the 45-term. We consider any one particular such routeing, which corresponds to the SU(N) fundamental representation path-ordered phase factros joining across the vertices to form a particular collection of closed loops, and we immediately see, that when we contract the upper-case indices through a vertex, the fields in each SU(N) fundamental representation path-ordered phase factor are in the *same* SU(N) subgroup of $(\mathrm{SU}(N))^M$ in both members of each pair of SU(N) fundamental representation path-ordered phase factors that join onto one another across a vertex. It immediately follows from this, that in each closed loop of SU(N) fundamental representation path-ordered phase factors, all the fields occurring in all the phase factors in that closed loop, are in the *same* SU(N) subgroup of $\mathrm{SU}(N))^M$. *Thus each closed loop of* SU(N) *fundamental representation path-ordered phase factors, (formed by joining* SU(N) *fundamental representation path-ordered phase factors in the individual paths, across the vertices, in accordance with the chosen routeing), is a Wilson loop in one of the* M SU(N) *subgroups of* $(\mathrm{SU}(N))^M$. We note, furthermore, that if any SU(N) fundamental representation Wilson loop passes along any new path *in both directions*, then both the SU(N) fundamental representation path-ordered phase factors passing along that path will be in the *same* one of the M SU(N)'s hence the $(1-\delta_{JK})$ factor in the 45-term for that path will vanish. Hence any routeing of the $(\mathrm{SU}(N))^M$ indices through the vertices which results in any of the SU(N) fundamental representation Wilson loops produced by that routeing, passing along any new path in both directions, gives no contribution at all, hence we may restrict the sum over routeings of the $(\mathrm{SU}(N))^M$ indices through the vertices to routeings which satisfy the "selection rule" that *none* of the SU(N) fundamental representation Wilson loops which they produce, passes along any new path in both directions.

2.2.1 Summing over partitions of the set of the SU(N) Wilson loops, into parts whose members belong to the same SU(N), subject to the selection rule

We now have to sum over the upper-case indices, (which distinguish the M separate SU(N) subgroups of $(\mathrm{SU}(N))^M$), subject to the constraint, which follows immediately from the foregoing, that all the upper-case indices around any SU(N) fundamental representation Wilson loop be equal, with their common value identifying the particular one of the M SU(N) subgroups of $(\mathrm{SU}(N))^M$ to which all the fields occurring in that SU(N) fundamental representation Wilson loop belong, and also to the constraint, which follows from the $(1 - \delta_{JK})$ factor in the 45-term, that *no new path has the two* SU(N) *fundamental representation path-ordered phase factors that pass along it, being in the* same SU(N) *subgroup of* $(\mathrm{SU}(N))^M$. Thus for each choice of the routeings through the vertices, we have to sum over all *partitions*, into not more than M parts, of the set of all the SU(N) fundamental representation closed loops that result from that routeing, subject to the constraint that no two loops that are members of the same part of the partition, pass along any common new path. And for each such partition, we sum over all distinct ways of assigning a distinct one of the M SU(N)'s to each part of the partition, (so that all the members of any given part of the partition, are in the *same* SU(N), which is different from the SU(N) assigned to every other part of the partition). We note that we are here considering together *all* the $\tilde{W}_i$'s that occur in the $(\mathrm{SU}(N))^M$ correlation function $[\tilde{W}_1 \dots \tilde{W}_s]_{(\mathrm{SU}(N))^M}$, *not* just one of the $\tilde{W}_i$'s on its own, and that the set of SU(N) fundamental representation Wilson loops that is being partitioned is the set of *all* the SU(N) Wilson loops that arise in the manner described, in *all* the $\tilde{W}_i$'s.

Now the $(\mathrm{SU}(N))^M$ correlation function $[\tilde{W}_1 \dots \tilde{W}_s]_{(\mathrm{SU}(N))^M}$ is itself equal to the sum, over all partitions of the set $\{\tilde{W}_1, \dots, \tilde{W}_s\}$, of $(-1)^{m-1}(m-1)!$, where m is the number of parts of the partition, times the product, over the parts of the partition, of the vacuum expectation value, in the Yang Mills theory for $(\mathrm{SU}(N))^M$, of the $\tilde{W}_i$'s in that part of the partition. (This is simply the inverse of the general expression (1.56) for vacuum expectation values in terms of correlation functions, and has already been used in Section 2.)

But the vacuum expectation value, in the Yang Mills theory for $(\mathrm{SU}(N))^M$, of any product of gauge-invariant quantities, each individual one of which involves just *one* of

the SU(N)'s, factorizes exactly into the product, over the individual SU(N)'s, of the vacuum expectation value, in that SU(N), of the product of all those factors in the original product of gauge-invariant quantities, that belong to that SU(N). Thus when we consider any term in our expansion of $[\tilde{W}_1 \dots \tilde{W}_s]_{(\mathrm{SU}(N))^M}$ as a sum of numerical coefficients times products of vacuum expectation values, in the Yang Mills theory for $(\mathrm{SU}(N))^M$, of subsets of the set $\{\tilde{W}_1, \dots, \tilde{W}_s\}$, (where each such product runs over the parts of some partition of $\{\tilde{W}_1, \dots, \tilde{W}_s\}$), and we also consider a particular term in our sum over all partitions of the set of all the SU(N) fundamental representation Wilson loops that have been generated from the paths in the $\tilde{W}_i$'s in the manner described, and a particular assignment of each part of *this* partition to a distinct one of the M SU(N)'s, (so that all the SU(N) Wilson loops belonging to any one part of this partition,are in the *same* SU(N), and SU(N) Wilson loops belonging to *distinct* parts of this partition, are in *distinct* SU(N)'s), we see that every term factorizes into a product of vacuum expectation values, in the Yang Mills theory for SU(N), of products of Wilson loops in the fundamental representation of SU(N).

2.2.2 No two Wilson loops, in any of the SU(N) vacuum expectation values, touch one another along any path

Furthermore, the $(1 - \delta_{JK})$ factor in the 45-term in (2.6) vanishes whenever two of our SU(N) fundamental representation Wilson loops that share a common "new" path, are in the *same* one of the M SU(N)'s. Hence, in every term in our double sum over partitions, (i.e. over all allowed partitions of the set of all our SU(N) fundamental representation Wilson loops, and over all partitions of the set $\{\tilde{W}_1, \dots, \tilde{W}_s\}$), any two of our SU($N$) fundamental representation Wilson loops that pass alon any common new path, lie in *different* SU(N) vacuum expectation values. *Hence, due to the* $(1 - \delta_{JK})$ *factor in the 45-term in (2.6), we* never *have two Wilson loops in any one of thes vacuum expectation values, touching one another along any path.* (We recall that, as already noted, the $(1 - \delta_{JK})$ factor also results in any routeing of the $(\mathrm{SU}(N))^M$ indices through the vertices that results in any of the SU(N) fundamental representation Wilson loops passingalong any new path in both directions, giving vanishing contribution, so that none of our individual SU(N) Wilson loops passes along any new path in both directions, hence none of our individual SU(N) Wilson loops touches itself along any path.) Furthermore, we see immediately that at any vertex at which three new paths

meet, the constraint that the two SU(N) fundamental representation path-ordered phase factors that run along each new path, must be in *different* SU(N)'s, means that *none* of the SU(N) vacuum expectation values in any of our products, contains more than one of the three SU(N) fundamental representation Wilson loops that meet at any such vertex. Thus in each vacuum expectation value in each of our products, the only "essential" intersection or touching, (or self-intersection or self-touching), of one or more of the SU(N) fundamental representation Wilson loops in that vacuum expectation value, that can occur, (where by "essential", we mean that is not a consequence of some accidental intersection or touching in the configuration space of the paths and vertices), is when two loops intersect, (or one loop intersects itself), at a quartic vertexin (1.31) or (1.23). And furthermore, this kind of "essential" intersection resulting from a quartic verex, is itself only an "occasional" occurrence, occurring either when the partition of the set of our SU(N) fundamental representation Wilson loops is such that two different loops that meet at some quartic vertex, are in the *same* part of that partition, (which results in two different loops in some vacuum expectation value intersecting one another), or else when the routeings of the $(\mathrm{SU}(N))^M$ indices through the vertices are such that some SU(N) fundamental representation Wilson loop intersects itself at some quaric vertex.

We thus see that, at least for the terms where we take the 45-term in (2.6) for every new path, an with the exception of the possible "essential" intersections which can arise from quartic vertices, (which will require special treatment in renormalization, but are not substantially worse than the "accidental" intersections which can occur due to intersections or coincidences of paths or vertices in configuration space), the right-hand sides of the group-changing equations for $(\mathrm{SU}(N))^M$ and SU(NM), applied to correlation functions, in the Yang Mills theory for SU(NM), of products of Wilson loops in the fundamental representantion of SU(NM), reduce to sums of products of vacuum expectation values, and hence, by (1.56), to sums of products of correlation functions, in the Yang Mills theory for SU(N), of products of Wilson loops in the fundamental representation of SU(N), and furthermore, the SU(N) fundamental representation Wilson loops that occur in any individual SU(N) vacuum expectation value or correlation function that occurs in the right-hand side have, in the absence of any accidental intersections or touchings associated with particular configurations in position space of the paths and vertices in the $\tilde{W}_i$'s, *no* intersections or touchings other than the possible "occasional" simple intersections described above arising from quartic

vertices. In other words, at least for all the terms where we choose the 45-term in (2.6) for every new path, all the complicated gauge-invariant "networks" with junctions that occur in the right-hand sides of the group-changing equations for general G and H, (even when we only have simple products of Wilson loops in the left-hand sides), have, in the present case of $G = (\mathrm{SU}(N))^M$ and $H = \mathrm{SU}(NM)$, with correlation functions of products of $\mathrm{SU}(NM)$ fundamental representation Wilson loop in the left-hand sides, *reduced away*, leaving only sums of products of correlation functions of products of $\mathrm{SU}(N)$ fundamental representation Wilson loops in the right-hand sides. We note that these results apply for *all* N.

2.2.3 Summing over assignments of the SU(N)'s to the parts of the partition gives a polynomial factor in M

We still have to sum over all the distinct ways of assigning a distinct one of the M $\mathrm{SU}(N)$ subgroups of $(\mathrm{SU}(N))^M$ to each of the r parts of our partition of the st of all our $\mathrm{SU}(N)$ fundamental representation Wilson loops, but we now see that this sum simply gives a factor

$$M(M-1)(M-2)\dots(M-r+1) = \frac{M!}{(M-r)!}$$

Now this is a polynomial in M that vanishes for $r \geq (M+1)$, hence we may extend the sum over the partitions of the set of all our $\mathrm{SU}(N)$ fundamental representation Wilson loops, (which was previously restricted to partitions with not more than M parts), into a sum over *all* partitions of the set of all our $\mathrm{SU}(N)$ fundamental representation Wilson loops, with the sum over all the distinct assignments of a distinct one of the M $\mathrm{SU}(N)$ subgroups of $(\mathrm{SU}(N))^M$ to each part of a partition into r parts, giving a factor

$$M(M-1)(M-2)\dots(M-r+1) = \frac{M!}{(M-r)!}$$

which vanishes for $r \geq (M+1)$. Hence we now see that, at least for the terms where we take the 45-term in (2.6) for every new path, the *only* dependence on M of the right-hand sides of our equations is through these simple factors $M(M-1)(M-2)\dots(M-r+1)$, there being precisely one such factor in each term, and that factor having its "r" equal to the number of parts of the particular partition of the set of all our $\mathrm{SU}(N)$ fundamental representation Wilson loops, that occurs in the definition of that term.

2.3 Inclusion of the 6-terms does not alter the main results

Let us now consider what happens when we choose the 6-term in (2.6), rather than the 45-term, in some of the new paths, We first note that there are *no* path-ordered phase factors in the 6-term, which is an immediate consequence of the fact that each 6-field interacts with *none* of the SU(N)'s. Furthermore, the t_6's are all diagonal matrices hence commute with one another, hence any SU(NM) structure constant with two or more of its indices in the set $\{(6E)|2 \leq E \leq M\}$, vanishes identically, hence at any vertex at which three new paths meet, *at most one of them* can get the 6-term rather than the 45-term in (2.6), while at any vertex at which four new paths meet, *at most two of them* can get the 6-term rather than the 45-term in (2.6). Furthermore, the $\delta_{JK}\delta_{jk}\delta_{PQ}\delta_{pq}$ index structure of the 6-term means that, as far as the index routeings and path-ordered phase factors are concerned, it is just as if any new path on which we choose the 6-term rather than the 45-term in (2.6), is simply not there at all. Thus we may completely analyse the effects of the 6-term in (2.6), by starting from "core" diagrams in which the 45-term in (2.6) is taken for every new path, and in which we have already specified, (for each particular "core" diagram term), what the index routeings through the vertices are, and what partition we take of the set of all the SU(N) fundamental representation Wilson loops produced by that routeing, and considering additions to each "core" diagram term, of additional new paths, each having both its ends on paths or vertices of the core diagram, and subject to the constraint that a total of at most four new paths can end at any vertex of the resulting diagram, with the 6-term in (2.6) being taken on each new path. Now f_{AaB} in (2.1) vanishes if either S or B is a "6" (i.e. a member of $\{(6E)|2 \leq E \leq M\}$), since every t_6 commutes with every t_1, t_2, and t_3, hence all the couplings of "6-paths", (i.e. new paths on which we take the 6-term in (2.6)), to vertices *not* on any of the original n SU(NM) fundamental representation Wilson loop closed paths, are via f_{ABC} in (2.2), with *one* of A, B, and C being a "6", and the other two being "45's", (i.e. members of $\{(4EeFf)|1 \leq E < F \leq M, 1 \leq e \leq N, 1 \leq f \leq N\}$ or $\{(5EeFf)|1 \leq E < F \leq M, 1 \leq e \leq N, 1 \leq f \leq N\}$). And we immediately see that when one of A, B, and C is a "6" and the other two are "45's", each of the two terms in the right-hand side of (2.2) corresponds, when a new "6-path" ends on a new path of the "core" diagram to form a new vertex at which two "45" paths, (i.e. new

paths on which we take the 45-term in (2.6)), and one "6" path end, to the upper-case indices "J" and "K" at that end of that "6" path, both being equal to one, or both being equal to the other, of the two different numbers wihch identify the two different SU(N)'s to whcih the two SU(N) fundamental representation path-ordered phase factors that run along that "45" path in the "core" diagram, belong, (in that particular term in our sum over all partitions of the set of all the SU(N) fundamental representation Wilson loops in that particular core diagram term). Now when a "6" path "attaches" to the *same* one of the M SU(N)'s at both its ends, the $\left(-\delta_{JQ}+\frac{1}{M}\right)$ factor in the 6-term in (2.6) takes the value $-\frac{(M-1)}{M}$, while when a "6" path "attaches" to a *different* SU(N) at each end, the $\left(-\delta_{JQ}+\frac{1}{M}\right)$ factor in the 6-term takes the value $\frac{1}{M}$. And whether a "6"-path "attaches" to the *same* SU(N) at each end or to a *different* SU(N) at each end is determined, firstly, by whether it "attaches" to the *same* SU(N) fundamental representation Wilson loop at each end or to a *different* SU(N) fundamental representation Wilson loop at each end, and secondly, if it attaches to a *different* SU(N) fundamental representation Wilson loop at each end, whether or not those two different SU(N) fundamental representation Wilson loops are members of the same, or different, parts of the partition of the set of *all* the SU(N) fundamental representation Wilson loops of the present particular "core" diagram term, to which the present "partition-term" corresponds. Hence, for each particular "partition-term" in each particular "core" diagram term, the *only* modification to the M-dependence of that term produced by the addition of a new "6"-path, is by multiplication by a factor $-\frac{(M-1)}{M}$ or $\frac{1}{M}$, as appropriate.

Furthermore, due to the absence of any path-ordered phase factor associated with any "6"-path, we can do the configuration-space sum over paths for any "6"-path *immediately*, to obtain simply a *free* Landau-gauge vector boson propagator or a *free* Fadeev-Popov propagator, as appropriate.

We must, however, be careful to note that, when we identify the individual $\tilde{W}_i$'s, (which are the "inseparable units" with respect to which the correlation function $[\tilde{W}_1 \dots \tilde{W}_s]_{(\mathrm{SU}(N))^M}$ is expressed in terms of $(\mathrm{SU}(N))^M$ vacuum expectation values by the inverse of (1.56)), as the connected parts of any decoration of our initial n SU(NM) fundamental representation Wilson loops, (as is required by the general derivation of the group-changing equations given in Section 2), we must consider *all* the new paths, i.e. all the "45" paths *and* all the "6" paths, in case two or more parts of the decoration, which would appear to be disconnected from one another if we only look at the

"45" paths, (i.e. if we only look at the "core" decoration, which contains all the "45" paths but none of the "6" paths), are in fact connected to one another by one or more "6" paths in the full decoration.

We should also note that when the full decoration contains some "6" paths, there will be cases when the Fadeev-Popov paths in the "core" decoration do not form closed lops, due to some of the paths in the Fadeev-Popov loops in the full decoration being "6" paths.

We also note here that, due to the $(1 - \delta_{JK})$ factor in the 45-term in (2.6), any *"core"* decoration whose number of connected components can be increased by the deletion of just *one* "45"-path, gives no contribution at all, since no matter what routeing of the indices through the vertices is chosen, the two SU(N) fundamental representation path-ordered phase factors that run (in opposite directions) along that "key" "45"-path, must be in the *same* SU(N) fundamental representation Wilson loop, (for otherwise, when the key "45"-path was deleted, any two points of the "core" diagram that were previously connected to one another via a sequence of paths that included the "key" "45"-path, would remain connected to one another by an alternative sequence of paths that replaced the "key" "45"-path in the original sequence of paths by a sequence of paths that traced the unbroken remainder part of either of the two different SU(N) fundamental representation Wilson loops that passed along the "key" "45"-path, so that deleting the "key" "45"-path would *not* increase the number of connected components of the "core" decoration, contrary to assumption), hence for every routeing of the indices through the vertices, the $(1 - \delta_{JK})$ factor for the "key" "45"-path, vanishes.

Thus, in summary, we see that our previous conclusion, namely that the right-hand sides of the group-changing equations for $(\mathrm{SU}(N))^M$ and SU(NM), applied to correlation functions, in the Yang Mills theory for SU(NM), of products of Wilson loops in the fundamental representation of SU(NM), reduce to sums of products of correlation functions, in the Yang Mills theory for SU(N), of products of Wilson loops in the fundamental representation of SU(N), and furthermore, that the SU(N) fundamental representation Wilson loops that occur in any individual SU(N) correlation function that occurs in the right-hand side, have, in the absence of any accidental intersections or touchings associated with particular configurations in position space of the paths and vertices in the $\tilde{W}_i$'s, *no* intersections or touchings, other than the possible "occasional" simple intersections arising from quartic vertices as described before, (these

conclusions being proved previously only for the case where we choose the 45-term in (2.6) for every new path), remain completely valid in the general case, where both the 45-term in (2.6) and the 6-term in (2.6) are allowed.

Furthermore, the entire M-dependence of the right-hand sides is given by simple coefficients, where the coefficient that applies for a given routeing of the indices through the vertices of the "core" decoration, (i.e. the decoration with all the "6"-paths removed), and a given partition P of the set of all the SU(N) fundamental representation Wilson loops that result from that routeing, is given by

$$M(M-1)(M-2)\dots(M-r+1) = \frac{M!}{(M-r)!}$$

where r is the number of parts of the partition P, and there is in addition, for each "6"-path, a factor of either $-\frac{(M-1)}{M}$ or $\frac{1}{M}$, where the factor for a given "6"-path is $-\frac{(M-1)}{M}$ if both ends of that "6"-path "attach" to the *same* SU(N) fundamental representation Wilson loop, or to two different SU(N) fundamental representation Wilson loops that are members of the same part of the partition P, and is $\frac{1}{M}$ if each end of that "6"-path "attaches" to a *different* SU(N) fundamental representation Wilson loop, and those two SU(N) fundamental representation Wilson loops are members of *different* parts of the partition P, where we note that for every "6"-path end that is *not* simply on one of the n original SU(NM) fundamental representation Wilson loops, we also have to sum over the two terms in the right-hand side of (2.2), (in the case where exactly one of A, B, and C is a "6", and the other two are "45's"), which corresponds, due to the $\delta_{JK}\delta_{jk}$ index structure at the $JjKk$ "end" of the 6-term in (2.6), to summing over two alternative choices of which of two alternative possible SU(N) fundamental representation path-ordered phase factor "lines" that "6"-path "attaches" to.

And we note, furthermore, that due to the absence of any path-ordered phase factor associated with any "6"-path, we can do the configuration-space sum over paths for any "6"-path *immediately*, to obtain simply a *free* Landau-gauge vector boson propagator or a *free* Fadeev-Popov propagator, as appropriate.

2.4 Closed equations for the expansion coefficients in the $\frac{1}{N}$ expansions of vacuum expectation values and correlation functions

We are now ready to apply the group-changing equations for $SU(N))^M$ and $SU(NM)$ to obtain a complete and closed set of equations among the expansion coefficients $f_r(W_1, \ldots, W_n, g^2)$ in (1.55).

Now if we were to take the Yang Mills action density for SU(NM) to be

$$\frac{NM}{4g^2} F_{\mu\nu\alpha} F_{\mu\nu\alpha}$$

(where α runs over all the generators of SU(NM)), then the expansions for the Wilson loop correlation functions of SU(NM) would be obtained from (1.55) simply by replacing N by NM. But the group-changing equations apply when the external factor of the action density is the *same* for both the "large group" H and the "small group" G, hence we take the action density as

$$\frac{N}{4g^2} F_{\mu\nu\alpha} F_{\mu\nu\alpha}$$

for *both* $H = \mathrm{SU}(NM)$ and $G = (\mathrm{SU}(N))^M$, with the only difference being that the sum on α runs over the generators of SU(NM) for SU(NM), and over the generators of $(\mathrm{SU}(N))^M$ for $\mathrm{SU}(N))^M$.

Hence the SU(NM) fundamental representation correlation functions, in the Yang Mills theory for SU(NM), are given in terms of the expansion coefficients $f_r(W_1, \ldots, W_n, g^2)$ in (1.55), by:

$$(NM)^{2-n} \left(f_0(W_1, \ldots, W_n, g^2 M) + \frac{1}{(NM)^2} f_1(W_1, \ldots, W_n, g^2 M) + \right.$$

$$\left. + \frac{1}{(NM)^4} f_2(W_1, \ldots, W_n, g^2 M) + \ldots \right) \tag{2.7}$$

where the replacement of the argument g^2 of the f_r's by $g^2 M$ is due to having the external factor $\frac{N}{4g^2}$ rather than the external factor $\frac{NM}{4g^2}$ in the Yang Mills action density for SU(NM).

We now proceed to derive closed equations for the expansion coefficients in (1.55) and (2.7), by substituting the expansions (2.7) into the left-hand sides, and the expansions (1.55) into the right-hand sides, of the group-changing equations for $(\mathrm{SU}(N))^M$

and SU(NM), applied to the correlation functions, in the Yang Mills theory for SU(NM), of n Wilson loops $W_1, \ldots, W_n$, each in the fundamental representation of SU(NM)

We need a Lemma which expresses the correlation functions of s "bunches" of quantities, (where in practice each "bunch" of quantities will be the product of all the SU(N) fundamental representation Wilson loops, in various SU(N) subgroups of $(\mathrm{SU}(N))^M$, that occur in a particular connected component $\tilde{W}_i$ of some decoration of our initial n Wilson loops, for a particular routeing of the $(\mathrm{SU}(N))^M$ indices through the vertices of that connected component of that decoration), in terms of the correlation functions of the individual quantities.

We recall [26] that a partition is a set P such that every member of P is a set, no member of P is empty, and if i and j are any two distinct members of P, then $i \cap j = \emptyset$, where $\emptyset$ is the empty set.

For any finite set X, we define $\mathbf{P}(X)$ to be the set whose members are all the partitions of X, (so that, in other words, $\mathbf{P}(X)$ is the set whose members are all the partitions P such that the union of all the members of P is equal to X), and we define $\mathbf{N}(X)$ to be equal to the number of members of X.

And we also recall [26] that if X is a set, and Y is a set whose members are sets, then we say that X is Y-connected if and only if, for every partition of X into two nonempty parts, there exists a member of Y that has nonempty intersection with both those parts, and we also recall that a Y-connected component of X is a nonempty Y-connected subset of X that is *not* a strict subset of any Y-connected subset of X. (We also recall our convention that the statement, "Z is a subset of X," includes the possibility that $Z = X$, and that we say, "Z is a strict subset of X," if we wish to exclude the possibility that $Z = X$.)

And we also recall [26] that if Y is a set whose members are sets, then we define $\mathcal{U}(Y)$ to be the union of all the members of Y.

2.4.1 A Lemma about correlation functions

Lemma Let K be a finite set of indices, (for example, the integers 1 to k), and let $\{W_i | i \in K\}$ be a set of quantities indexed by K, (for example, $W_1, \ldots, W_k$ could be a set of k Wilson loops.) Let S be a partition of K.

For each member i of S we define:

$$B_i = \prod_{j \in i} W_j \tag{2.8}$$

Then the correlation function of the B_i's, in which each B_i is treated as an "indivisible unit", namely:

$$\begin{aligned}\left[\prod_{i \in S} B_i\right] &= \sum_{P \in \mathbf{P}(S)} (-1)^{(\mathbf{N}(P)-1)} (\mathbf{N}(P)-1)! \prod_{i \in P} \left\langle \prod_{j \in i} B_j \right\rangle \\ &= \sum_{P \in \mathbf{P}(S)} (-1)^{(\mathbf{N}(P)-1)} (\mathbf{N}(P)-1)! \prod_{i \in P} \left\langle \prod_{j \in i} \prod_{l \in j} W_l \right\rangle \\ &= \sum_{P \in \mathbf{P}(S)} (-1)^{(\mathbf{N}(P)-1)} (\mathbf{N}(P)-1)! \prod_{i \in P} \left\langle \prod_{l \in \mathcal{U}(i)} W_l \right\rangle \end{aligned} \tag{2.9}$$

is equal to a sum:

$$\sum_R \prod_{i \in R} \left[\prod_{j \in i} W_j\right] \tag{2.10}$$

where the sum runs over all partitions R of K such that K is $(R \cup S)$-connected.

Proof. We substitute into the right-hand side of (2.9) the general identities (1.56), which in our present notation, are expressed as:

$$\left\langle \prod_{i \in N} W_i \right\rangle = \sum_{Q \in \mathbf{P}(N)} \prod_{i \in Q} \left[\prod_{j \in i} W_j\right] \tag{2.11}$$

for a general finite set N, to obtain the correlation function of the B_i's in terms of the correlation functions of the W_j's as:

$$\sum_{P \in \mathbf{P}(S)} (-1)^{\mathbf{N}(P)-1)} (\mathbf{N}(P)-1)! \prod_{i \in P} \sum_{Q \in \mathbf{P}(\mathcal{U}(i))} \prod_{j \in Q} \left[\prod_{l \in j} W_l\right] \tag{2.12}$$

This has the general form of (2.10), namely a sum of products of correlation functions of the W_j's, with each individual product corresponding to a partition of K, (in the sense that the individual correlation-function factors in that product are in one-to-one correspondence with the parts of that partition of K), and the partition R of K that corresponds to a particular partition P of S, and particular partitions Q_i, (one for each part i of P), of the unions of the parts of P, being given by:

$$R = \bigcup_{i \in P} Q_i \tag{2.13}$$

It remains to determine the net coefficient of the product corresponding to each partition R of K, and check that it is equal to 1 if K is $R \cup S$-connected, and equal to 0 otherwise.

We begin by asking, for a general partition R of K, what partitions P of S can produce R by (2.13), for suitable partitions Q_i of the unions of the parts i of P, and we see immediately that a partition P of S is able to produce R in this way if and only if, for every member i of P and every member j of R, j is either a *subset* of $\mathcal{U}(i)$, or else $j \cap \mathcal{U}(i)$ is empty, or in other words, a partition P of S is able to produce R in this way if and only if, for every member j of R, j intersects the union of *exactly one* member i of P, (and hence is a subset of the union of that member i of P). And furthermore, if a partition P of S *is* able to produce R in this way, then there is *exactly one* partition Q_i of the union of each member i of P that produces R in this way from P: namely, for each member i of P, Q_i is the set of all the members of R that are subsets of $\mathcal{U}(i)$.

Thus for a general partition R of K, the net coefficient in (2.12) of the product corresponding to R, is equal to a sum:

$$\sum_P (-1)^{(\mathbf{N}(P)-1)} (\mathbf{N}(P)-1)! \tag{2.14}$$

where the sum runs over all partitions P of S such that each member of R intersects the union of exactly one member of P, (and hence is a subset of the union of that member of P).

We next note that the set of all partitions P of S such that each member of R intersects the union of exactly one member of P, (and hence is a subset of the union of that member of P), is in one-to-one correspondence with the set of all partitions T of the set of all the $R \cup S$-connected components of K, and that furthermore, each such partition P of S has the same number of members as the corresponding partition T of the set of all the $R \cup S$-connected components of K. For if P is any such partition of S, and j is any $R \cup S$-connected component of K, then j is certainly a subset of the union of some member of P, since every member of S is a subset of the union of some member of P, and every member of R is a subset of the union of some member of P. (We recall [26] that the set of all the $R \cup S$-connected components of K, is itself a partition of K.) And in fact, if P is any such partition of S, and we define the corresponding partition U *of* K to be the set whose members are the unions $\mathcal{U}(i)$ of the members i of P, then the corresponding partition T of the set of all the $R \cup S$-connected components of K, is the set whose members are in one-to-one correspondence with the members of U,

and such that the member of T that corresponds to a given member m of U, is the set whose members are all the $R \cup S$-connected components of K that are subsets of m, (where we note that there is at least one such $R \cup S$-connected component of K, since m is nonempty, and the set of all the $R \cup S$-connected components of K is a partition of K such that each individual $R \cup S$-connected component of K is either a subset of m or else does not intersect m). And given any partition T of the set of all the $R \cup S$-connected components of K, we immediately obtain the corresponding "allowed" partition P of S by reversing this construction, and we note that each of the three corresponding partitions P, U, and T has the same number of members. Hence for any partition R of K, the net coefficient in (2.12) of the product of correlation functions of the W_j's that corresponds to R, is given by:

$$\sum_{T \in \mathbf{P}(V)} (-1)^{(\mathbf{N}(T)-1)} (\mathbf{N}(T) - 1)! \tag{2.15}$$

where we have defined V to be the set of all the $R \cup S$-connected components of K.

Now the sum (2.15) is well known to have the value 1 if $\mathbf{N}(V) = 1$, and the value 0 if $\mathbf{N}(V) \geq 2$, and indeed, this very same sum, with V replaced by a general finite set N, is exactly what occurs in the verification fo the standard inversion formula:

$$\left[\prod_{i \in N} W_i\right] = \sum_{Q \in \mathbf{P}(N)} (-1)^{(\mathbf{N}(Q)-1)} (\mathbf{N}(Q) - 1)! \prod_{i \in Q} \left\langle \prod_{j \in i} W_j \right\rangle \tag{2.16}$$

of the standard expression (2.11) for the vacuum expectation values in terms of the correlation functions,(i.e. when (2.11) is substituted into the right-hand side of (2.16)).

This concludes the proof of the Lemma, for we have now shown that the coefficient in (2.11) of the product of correlation functions of the W_j's that corresponds to a partition R of K, is equal to 1 if R is such that K has exactly *one* $R \cup S$-connected component, (or in other words, if R is such that K is $R \cup S$-connected), and equal to 0 if R is such that K has *more* than one $R \cup S$-connected component.

For completeness, we note here that the fact that (2.15) is equal to 1 if $\mathbf{N}(V) = 1$, and equal to 0 if $\mathbf{N}(V) \geq 2$, may be proved by classifying the partitions T of V by their numbers m_q, $q \geq 1$, of q-membered parts, subject to the constraint that:

$$m_1 + 2m_2 + 3m_3 + \ldots = \mathbf{N}(V) \tag{2.17}$$

there being:

$$(\mathbf{N}(V))! \prod_{q \geq 1} \frac{1}{(q!)^{m_q} m_q!} \tag{2.18}$$

such partitions T of V. It is then plain that the value of (2.15) depends only on $\mathbf{N}(V) \equiv v$. We multiply (2.15) by $\frac{\lambda^v}{v!}$ and sum from $v = 0$ to ∞ to get:

$$\sum_{\substack{m_1 \geq 0 \\ m_2 \geq 0 \\ \cdots}} (-1)\left((m_1 + m_2 + m_3 + \ldots) - 1\right)! \prod_{q \geq 1} \frac{1}{m_q!} \left(\frac{-\lambda^q}{q!}\right)^{m_q} =$$

$$= \sum_{z=0}^{\infty} \frac{(-1)}{z} \sum_{\substack{m_1 \geq 0 \\ m_2 \geq 0 \\ \cdots \\ | \\ m_1+m_2+m_3+\ldots=z}} z! \prod_{q \geq 1} \frac{1}{m_q!} \left(\frac{-\lambda^q}{q!}\right)^{m_q} = \sum_{z=0}^{\infty} \frac{(-1)}{z} \left(\sum_{q=1}^{\infty} \left(\frac{-\lambda^q}{q!}\right)\right)^z =$$

$$= \sum_{z=0}^{\infty} \frac{(-1)}{z} \left(1 - e^{\lambda}\right)^z = \ln\left(1 - \left(1 - e^{\lambda}\right)\right) = \lambda \tag{2.19}$$

2.4.2 Substitution of the $\frac{1}{N}$ expansions into the Group-Changing Equations for $\mathbf{SU}(NM)$ and $(\mathbf{SU}(N))^M$

We now substitute the expansions (2.7) into the left-hand sides, and the expansions (1.55) into the right-hand sides, of the group-changing equations for $(\mathrm{SU}(N))^M$ and $\mathrm{SU}(NM)$, applied to the correlation functions,in the Yang Mills theory for $\mathrm{SU}(NM)$, of n Wilson loops $W_1, \ldots, W_n$, each in the fundamental representation of $\mathrm{SU}(NM)$, and equate coefficients of powers of N, in order to derive closed equations for the f_r's in (1.55) and (2.7), bearing in mind the reduction of the right-hadn sides of the group-changing equations for this case, to sums of products of correlation functions of $\mathrm{SU}(N)$ fundamental representation Wilson loops, in various $\mathrm{SU}(N)$ subgroups of $(\mathrm{SU}(N))^M$, as derived in the foregoing.

We adopt the following procedure. We write out the right-hand side of the group-changing equation for $[W_1 \ldots W_n]_{\mathrm{SU}(NM)}$ as a sum of decorations of the n closed paths defining the n Wilson loops $W_1, \ldots, W_n$, as given by the general group-changing equations as derived in Section 2. We consider a particular term $[\tilde{W}_1 \ldots \tilde{W}_s]_{(\mathrm{SU}(N))^M}$ in this sum over decorations, where $\tilde{W}_1, \ldots, \tilde{W}_s$ are the connected components of this particular decoration of $W_1, \ldots, W_n$, (and some of the $\tilde{W}_i$'s may be vacuum bubbles). We also consider a specific assignment of which of the "new paths" of the decorations

$\tilde{W}_1, \ldots \tilde{W}_s$, (i.e. paths that are not wholes or parts of any of the n original closed paths), are gauge-field paths, (to be summed over with kinematic weights given by (1.16), (1.18), (1.20), and (1.27), and which are Fadeev-Popov paths, (to be summed over with kinematic weights given by (1.24) and (1.27)), consistent with the "action vertices" as given by (1.31), and the second and third terms in (1.23), in consequence of which the Fadeev-Popov paths form closed loops, which involve new paths and action vertices only. We note that, as usual, there is a factor (-1) for each closed loop of Fadeev-Popov paths, corresponding to the Fermi statistics of the Fadeev-Popov fields. We also note that the gauge-covariant derivatives in the second term in (1.23) and the first term in (1.31) result in the modification of the kinematic weights of the appropriate paths by the addition of an extra segment to each term in (1.27), with a weight fro this segment as given by(1.29). We also note that, as mentioned between equations (2.1) and (2.2), at each vertex where a new path ends at one of the n original closed paths, we have either a t_4, a t_5, or a t_6, and we note that any new path that ends at one of the n original closed paths, is a gauge-field path, not a Fadeev-Popov path, (since the definition (1.1) of a path-ordered phase factor involve no Fadeev-Popov fields).

We call a path in a decoration an "original" path if it is the whole, or a part, of any of the n original closed paths, and a "new" path if it is not an original path, or in other words, if it is a gauge-field path or a Fadeev-Popov path. And we call a vertex in a decoration an "action" vertex if it arises from a term in (1.31) or from the second or third term in (1.23), and an "original-path vertex" if it is a vertex where a gauge-field path ends at one of the n original closed loops.

We also consider a specific assignment of which of the new paths are "45"-paths and which are "6"-paths, subject to the selection rules, derived in the foregoing, that at most one 6-path can end at any cubic action vertex, and at most two 6-paths can end at any quartic action vertex, and that it must not be possible to increase the number of connected components of the "core" part of any $\tilde{W}_i$, (i.e. of that $\tilde{W}_i$ with all its 6-paths removed), by removing just *one* 45-path. (We note that it *is* possible, however, for the core part of a $\tilde{W}_i$ to have more than one connected component.)

We express all the structure constants in the action vertices, (i.e. in (1.31), and the second and third term in (1.23)), in terms of traces of the t_i's ($1 \leq i \leq 6$), by (2.1) and (2.2), and consider a specific choice, independently for each structure constant in each vertex, of one of the two terms in (2.1) or (2.2).

We also consider separately, at each quartic action vertex at which four gauge-field

45-paths meet, the term

$$\frac{1}{4g^2} f_{aBC} f_{aEF} A_{\mu B} A_{\nu C} A_{\mu E} A_{\nu F}$$

in (1.31), (which corresponds to a being summed over the 1's, the 2's, and the 3's), and the part of the

$$\frac{1}{4g^2} f_{ABC} f_{AEF} A_{\mu B} A_{\nu C} A_{\mu E} A_{\nu F}$$

term in (1.31) where A is summed over the 4's and the 5's, and the part of the

$$\frac{1}{4g^2} f_{ABC} f_{AEF} A_{\mu B} A_{\nu C} A_{\mu E} A_{\nu F}$$

term in (1.31) where A is summed over the 6's.

We note that at any quartic action vertex where one or two 6-paths end, all four new paths ending at that vertex must be gauge-field paths, since in the

$$\psi_A f_{AaB} A_{\mu B} \phi_C f_{CaD} A_{\mu D}$$

in (1.23), the sum on a runs only over the 1's, the 2's, and the 3's, and f_{AaB} and f_{CaD} vanish when any of their upper-case indices is a "6", and furthermore the only term in (1.31) that contributes in these cases is the

$$f_{ABC} f_{AEF} A_{\mu B} A_{\nu C} A_{\mu E} A_{\nu F}$$

term, with the sum on A running just over the 4's and the 5's, Since f_{ABC} vanishes when two or more of its indices are 6's, and likewise for f_{AEF}.

We use (1.59), (1.60), (1.61), and (2.6), to express the particular term we are considering, as a sum of a finite number of terms, each of which is a product of SU(N) fundamental representation Wilson loops, with the individual Wilson loops being in various SU(N) subgroups of $(\mathrm{SU}(N))^M$, and where the sum is over the distinct allowed assignments of the individual Wilson loops to the different SU(N) subgroups of $(\mathrm{SU}(N))^M$, subject to the constraint, (due to the $(1 - \delta_{JK})$ factor in the 45-term in (2.6)), that any two Wilson loops that share a common 45-path, must be in *different* SU(N) subgroups of $(\mathrm{SU}(N))^M$. (There is also an analogous constraint, coming from the $(1 - \delta_{JK})$ factor in (1.60), at each quartic vertex at which four gauge-field 45-pats meet, and at which we have chosen the $f_{ABC} f_{AEF} A_{\mu B} A_{\nu C} A_{\mu E} A_{\nu F}$ term in (1.31), with A being summed over the 4's and the 5's.)

We note that, at any quartic vertex that corresponds to the $f_{aBC}f_{aEF}A_{\mu B}A_{\nu C}A_{\mu E}A_{\nu F}$ term in (1.31) or to the $\psi_A f_{AaB}A_{\mu B}\phi_C f_{CaD}A_{\mu D}$ term in (1.23), two of the four SU(N) fundamental representation path-ordered phase factors that pass through the vertex are forced, by (1.59), to be in the *same* SU(N) subgroup of $(\mathrm{SU}(N))^M$.

(We note that the $\delta_{JA}\delta_{KA}\delta_{PA}\delta_{QA}$ factor in (1.59), when summed over A, gives $\delta_{JQ}\delta_{PK}\delta_{JK}$, which is also equal to $\delta_{JK}\delta_{PQ}\delta_{JQ}$, and that the summation convention is of course *not* to be applied to J, K, or Q here.) We also consider separately, at every such quartic vertex, the $-\delta_{JA}\delta_{KA}\delta_{PA}\delta_{QA}\delta_{jq}\delta_{kp}$ and the $\frac{1}{N}\delta_{JA}\delta_{KA}\delta_{PA}\delta_{QA}\delta_{jk}\delta_{pq}$ term in the right-hand side of (1.59), (making an independent choice of one or the other of these two terms at every such quartic vertex).

Now our choice, independently at each cubic action vertex, of one of the two terms in (2.2), and independently at each structure constant in each quartic action vertex, of one of the two terms in (2.1) or in (2.2), as appropriate, and independently, at each quartic vertex at which four gauge-field 45-paths meet, of either the "123-terms", the "45-terms", or the "6-terms" in $(f_{aBC}f_{aEF} + f_{ABC}f_{AEF})$ in (1.31), and independently, at each "123-type" quartic vertex, (i.e. the $f_{aBC}f_{aEF}$ term in (1.31) or the quartic interaction term in (1.23)), of either the $-\delta_{jq}\delta_{kp}$ term or the $\frac{1}{N}\delta_{jk}\delta_{pq}$ term in (1.59), corresponds to making a specific choice, independently at each vertex, of the routeings of all the SU(N) fundamental representation path-ordered phase factors that pass through that vertex, and hence to a specific choice of the way that the pairs of (oppositely-directed) SU(N) fundamental representation path-ordered phase factors that run alon the 45-path, and the single SU(N) fundamental representation path-ordered phase factors that run along each original path, are connected up into SU(N) fundamental representation Wilson loops, and it also corresponds, wherever a 6-path ends at an action vertex,(and hence at a structure constant), rather than at one of the n original closed paths, to making a specific choice of which of the two SU(N) fundamental representation path-ordered phase factors that "pass through" that structure constant, (when (2.6) or (1.60) is applied to each of the other two indices of that structure constant, each of which must be either a "4" or a "5"), that 6-path end "attaches to", in the sense that the two upper-case indices at that end of that 6-path, are both equal to the integer A, $1 \leq A \leq M$, which specifies to which of the M SU(N) subgroups of $(\mathrm{SU}(N))^M$, the SU(N) fundamental representation path-ordered phase factor to which that end of that 6-path "attaches", belongs. (For when that structure

constant is expressed, by (2.2), as:

$$f_{ABC} = -\left((t_A)_{LlJj}\,(t_B)_{JjKk}\,(t_C)_{KkLl} - (t_C)_{LlJj}\,(t_B)_{JjKk}\,(t_A)_{KkLl}\right) \tag{2.20}$$

we find, if B is the "6", and A and C are the 4's and 5's, that when we contract with $(t_B)_{PpQq}$, say, with B being summed over all the "6's", and use (2.6), remembering that the 6-term in the right-hand side of (2.6) is the contribution of all the terms in the left-hand side that involve one or more t_6's, and that the only nonvanishing such term in the left hand side is precisely $\delta_{AB}\,(t_A)_{JjKk}\,(t_B)_{PpQq}$, with A and B being summed over all the "6's", we get:

$$-\left((t_A)_{LlJj}\,(t_C)_{KkLl} - (t_C)_{LlJj}\,(t_A)_{KkLl}\right)\frac{1}{N}\delta_{JK}\delta_{jk}\delta_{PQ}\delta_{pq}\left(-\delta_{JQ}+\frac{1}{M}\right) \tag{2.21}$$

where J, K, and L are to be summed from 1 to M, and j, k, and l are to be summed from 1 to N, and this is equal to:

$$-\,(t_A)_{LlJj}\,(t_C)_{JjLl}\,\frac{1}{N}\delta_{PQ}\delta_{pq}\left(\left(-\delta_{JQ}+\frac{1}{M}\right)-\left(-\delta_{LQ}+\frac{1}{M}\right)\right) \tag{2.22}$$

where J and L are to be summed from 1 to M, and j and l are to be summed from 1 to N. The stated result follows from this if we note that if we multiply by $W((\mathrm{SU}(N))^M, x(s))_{EA}\,(t_E)_{GgHh}$, say, sum E and A over the 4's and 5's, and use the "45 part" of (2.6), then J and L, (which become equal to G and H, respectively, by the δ_{GJ} and δ_{LH} factors in the 45-term in the form of (2.6) with the appropriate indices), identify the two (distinct) $\mathrm{SU}(N)$ subgroups of $(\mathrm{SU}(N))^M$ to which the gauge fields occurring in the two $\mathrm{SU}(N)$ fundamental representation path-ordered phase factors which run, in opposite directions, along the path $x(s)$, belong.)

Chapter 3

The Group-Variation Equations for the SU(N) Groups

3.1 The Group-Variation Equations to all orders in $\frac{1}{N}$

3.1.1 Diagram notation

We introduce the following diagram notation for identifying individual terms of the type we have described. 45-paths will be denoted by a pair of parallel lines, representing the two SU(N) fundamental representation path-ordered phase factors which run, in opposite directions, along them. We adopt the rule, "drive on the left", for specifying the directions of the two phase factors, and arrows will only be put on the lines when their directions cannot be unambiguously identified by this rule. 6-paths will be indicated by single dashed lines. Original paths, (i.e. paths that form parts or wholes of any of the n original closed paths), will be indicated by single solid lines. The main purpose of the diagrammatic notation is to display the routeings of the SU(N) fundamental representation path-ordered phase factors through the vertices. We also indicate, wherever a 6-path ends at an action vertex, (and hence at a structure constant), which of the two SU(N) fundamental representation path-ordered phase factors passing through that structure constant, that 6-path end "attaches to", in the sense explained above, by ensuring that the broken line that represents the 6-path, ends with a dash that terminates on the appropriate SU(N) fundamental representation

path-ordered phase factor.

The possible triple action vertices are:

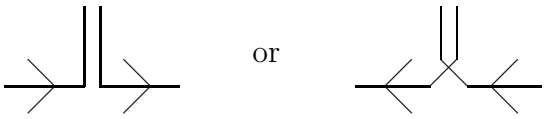
(3.1)

and:

(3.2)

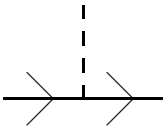
There is always a relative minus sign between the two terms in (3.1), corresponding to the relative minus sign between the two terms in (2.2), and there is similarly always a relative minus sign between the two terms in (3.2), again corresponding to the relative minus sign between the two terms in (2.2).

The original-path vertices are:



(3.3)

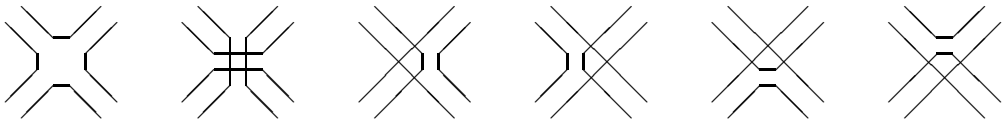
where the cross-over in the second form of (3.3) occurs simply in order to follow the rule: "drive on the left", (and we note that the two forms in (3.3) are two different ways of drawing the same thing, *not* two different terms to be added or subtracted from one another), and :



(3.4)

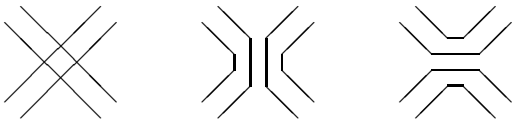
There are two distinct types of phase-factor routeing through quartic action vertices at which four 45-paths end:

Type 1



(3.5)

Type 2



(3.6)

Type 1 quartic vertices arise from the "45-terms" in $f_{ABC}f_{ADE}$ in (1.31), (when (1.60) is used), and from the $-\delta_{jq}\delta_{kp}$ term in (1.59), when (1.59) is applied to the

"123-terms", namely $f_{aBC}f_{aEF}$ in (1.31), and the quartic interaction term $\psi_A f_{AaB} A_{\mu B} \phi_C f_{CaD} A_{\mu D}$ in (1.23).

Type 2 quartic vertices arise from the "6-terms" in $f_{ABC}f_{ADE}$ in (1.31), (when (1.61) is used), and from the $\frac{1}{N}\delta_{jk}\delta pq$ term in (1.59), when (1.59) is applied to the "123-terms".

Type 1 quartic vertices are characterized by the property that in two of the three "2/2 channels", (i.e. the three channels, traditionally called s, t, and u, in which the four legs of the quartic vertex are divided into two sets of two legs each), *two* of the four SU(N) fundamental representation path-ordered phase factors that pass through that vertex, pass from one to the other of the two sets of two legs that define that channel, (or in other words, "pass through that channel"), while in the third 2/2 channel, all four path-ordered phase factors pass through that channel, and the Type 2 quartic vertices are characterized by the property that, in one of the three 2/2 channels, *none* of the path-ordered phase factors pass throught that channel, while in the other two 2/2 channels, all four of the path-ordered phase factors pass through those channels.

The other two types of quartic action vertex, which have either one or two 6-paths ending at them, and which we call Type 3 and Type 4 respectively, (and which arise only from the $f_{ABC}f_{AEF}$ term in (1.31), and not at all from (1.23), due to the vanishing of f_{aBC} whenever B or C is a "6"), look respectively like an (3.1) and an (3.2) in close proximity, or like two (3.2)'s in close proximity.

To have each diagram formed from these components correspond to precisely one term of the type described above, it would be necessary, for each Type 1 quartic vertex, to indicate whether that Type 1 quartic vertex arises from a "123-term" via equation (1.59), or from a "45-term" via equation (1.60), and also to indicate which of the two 2/2 channel through which *two* phase factors pass, corresponds to the "source" of that vertex, in the sense that there is one of the two structure constants from which that vertex arises, at each "end" of that channel, and it would also be necessary, for each Type 2 quartic vertex, to indicate whether that Type 2 quartic vertex arises from a "123-term" via equation (1.59), or from a "6-term" via equation (1.61), and also to indicate which of the four possible pairs of phase factors, (there being one member of each pair at each end of the "source" channel, which in this case is the one through which *none* of the phase factors pass), is the pair "involved" in that term, in the sense that whether those two phase factors are in the same, or different, SU(N) subgroups of $(\mathrm{SU}(N))^M$, determines whether or not the contribution from (1.59) vanishes, and

whether the contribution from (1.61) includes the factor $\frac{1}{M}$ or the factor $-\frac{(M-1)}{M}$.

Certain simplifications occur. Firstly, if it is known that two of the paths that meet at a quartic vertex are Fadeev-Popov paths, rather than gauge-field paths, (which is not shown on our diagrams), then that vertex can only arise from a "123-term", namely the term $\psi_A f_{AaB} A_{\mu B} \phi_C f_{CaD} A_{\mu D}$ in (1.23). And secondly, if it is known that all four paths that meet at a quartic vertex are gauge-field paths, (so that that vertex arises from the $(f_{aBC} f_{aEF} + f_{ABC} f_{AEF})$ term in (1.31)), then if that vertex is a Type-1 quartic vertex, we may add the contributions from (1.59) and from (1.60), to obtain the $-\delta_{JQ}\delta_{jq}\delta_{KP}\delta_{kp}$ term in (1.62), which is completely independent of which SU(N) subgroups of $(\mathrm{SU}(N))^M$ the four phase factors are in, (i.e. independent of whether or not any of them are in the same SU(N) subgroups of $(\mathrm{SU}(N))^M$), and if that vertex is a Type-2 quartic vertex, we may add the contributions from (1.59) and from (1.61) to obtain the term $\frac{1}{NM}\delta_{JK}\delta_{jk}\delta_{PQ}\delta_{pq}$ in (1.62), which is again completely independent of which SU(N) subgroups of $(\mathrm{SU}(N))^M$ the four phase factors are in, and which consequently *cancels out* when we sum over the four possible pairs of phase factors which can be "involved", due to the relative minus sign between the two terms in (2.1) and in (2.2). (This corresponds to a similar cancellation, in the Feynman diagrams for SU(N) Yang Mills theory, of the contributions from the $\frac{1}{N}\delta_{jk}\delta_{pq}$ term in (1.50) whenever that term occurs between two structure constants, i.e. inside a quartic vertex or between any two action vertices. Indeed, if the $\frac{1}{N}\delta_{jk}\delta_{pq}$ term in (1.50) is substituted for $(t_\alpha)_{jk}(t_\alpha)_{pq}$ in:

$$\begin{aligned} &\mathrm{tr}\,(t_\alpha t_\beta t_\gamma - t_\gamma t_\beta t_\alpha)\,\mathrm{tr}\,(t_\alpha t_\epsilon t_\phi - t_\phi t_\epsilon t_\alpha) = \\ &= (t_\beta t_\gamma - t_\gamma t_\beta)_{kj}\,(t_\alpha)_{jk}\,(t_\alpha)_{pq}\,(t_\epsilon t_\phi - t_\phi t_\epsilon)_{qp} \end{aligned} \tag{3.7}$$

we immediately obtain $\frac{1}{N}\mathrm{tr}\,(t_\beta t_\gamma - t_\gamma t_\beta)\,\mathrm{tr}\,(t_\epsilon t_\phi - t_\phi t_\epsilon) = 0$.)

But it is still necessary to indicate, for each Type-1 quartic vertex, which of the two 2/2 channels through which two phase factors pass, is the "source" of that quartic vertex.

However for our immediate purpose, namely to determine, for each f_r in the expansions (1.55) and (2.7), which of the f_s's can occur in the right-hand side of the group-changing equation for that f_r, there is no need to include any more detail in our diagrams than is given by the diagram rules as defined above. The important point to note is that, as follows immediately from (1.59) and (1.61), *there is an explicit factor of $\frac{1}{N}$ associated with every Type-2 quartic vertex.* We note here that, as follows

directly from the 6-term in (2.6), *there is also an explicit factor of* $\frac{1}{N}$ *associated with each 6-path.* We also note here that, as pointed out previously, the $(1-\delta_{JK})$ factor in the 45-term in (2.6) has the consequence, that any routeing of the phase factors through the vertices, that disobeys the "selection rule" that no SU(N) fundamental representation Wilson loop may pass in both directions along any 45-path, gives no contribution at all.

3.1.2 Application of the Lemma

Now the break up of the $\tilde{W}_i$'s into sums of terms that correspond to specific routeings of the SU(N) fundamental representation path-ordered phase factors through the vertices, operates completely independently within each separate $\tilde{W}_i$ in the $(\mathrm{SU}(N))^M$ correlation function $\left[\tilde{W}_1 \dots \tilde{W}_s\right]_{(\mathrm{SU}(N))^M}$, and consequently this $(\mathrm{SU}(N))^M$ correlation function is equal to a sum of $(\mathrm{SU}(N))^M$ correlation functions of the form $[B_1 \dots B_s]_{(\mathrm{SU}(N))^M}$, where each B_i is a term in the corresponding $\tilde{W}_i$ that corresponds to specific routeings through the vertices of the SU(N) fundamental representation path-ordered phase factors that occur in that $\tilde{W}_i$, and consequently to a specific set K_i of SU(N) fundamental representation Wilson loops, in various SU(N) subgroups of $(\mathrm{SU}(N))^M$, in that $\tilde{W}_i$, and also to specific attachment points of 6-path ends, where appropriate.

Furthermore, the assignment of the SU(N) fundamental representation Wilson loops in the B_i's to various SU(N) subgroups of $(\mathrm{SU}(N))^M$, subject to the selection rules explained in the foregoing, and in particular to the crucial selection rule that *no two* SU(N) *fundamental representation Wilson loops that pass, (in opposite directions), along a common 45-path, can be in the same* SU(N) *subgroup of* $(\mathrm{SU}(N))^M$, (which again follows directly from the $(1-\delta_{JK})$ factor in the 45-term in (2.6)), again operates completely independently within the separate B_i's in the $(\mathrm{SU}(N))^M$ correlation function $[B_1 \dots B_s]_{(\mathrm{SU}(N))^M}$, hence each such $(\mathrm{SU}(N))^M$ correlation function $[B_1 \dots B_s]_{(\mathrm{SU}(N))^M}$ is itself equal to a sum of $(\mathrm{SU}(N))^M$ correlation functions $\left[\tilde{B}_1 \dots \tilde{B}_s\right]_{(\mathrm{SU}(N))^M}$, where each $\tilde{B}_i$ corresponds to a specific assignment of the SU(N) fundamental representation Wilson loops in the corresponding B_i, to various specific SU(N) subgroups of $(\mathrm{SU}(N))^M$, consistent with the selection rules as described above.

For any such term we define, for each $\tilde{B}_i$, K_i to be the set of all the SU(N) fun-

damental representation Wilson loops in $\tilde{B}_i$, and S to be the set whose members are all the K_i's, (so that S is a partition of the set K of *all* the SU(N) fundamental representation Wilson loops in all the $\tilde{B}_i$'s). We note that $s = \mathbf{N}(S)$.

We now apply our Lemma to each such $(\mathrm{SU}(N))^M$ correlation function $\left[\tilde{B}_1 \dots \tilde{B}_s\right]_{(\mathrm{SU}(N))^M}$, to express it as a sum of produts of $(\mathrm{SU}(N))^M$ correlation functions of the individual SU(N) fundamental representation Wilson loops that occur in the $\tilde{B}_i$'s, as in (2.10), where the sum on R in (2.10) runs over all partitions R of the set K of all the SU(N) fundamental representation Wilson loops in all the $\tilde{B}_i$'s, such that K is $(R \cup S)$-connected. We note here that the requirement that R be such that K is $(R \cup S)$-connected, has the consequence that *the maximum possible number of parts of the partition R is equal to $k + 1 - s$, where $k = \mathbf{N}(K)$ is the total number of* SU(N) *fundamental representation Wilson loops in all the $\tilde{B}_i$'s.*

Now an $(\mathrm{SU}(N))^M$ correlation function $\left[\prod_{j \in i} W_j\right]_{(\mathrm{SU}(N))^M}$, where i is a part of the partition R, *vanishes* unless all the W_j's, $j \in i$, are in the *same* SU(N) subgroup of $(\mathrm{SU}(N))^M$, and we thus obtain the following additional selection rule on the assignments of the SU(N) fundamental representation Wilson loops to various SU(N) subgroups of $(\mathrm{SU}(N))^M$, for each partition R, as generated by the Lemma, of the set K of all the SU(N) fundamental representation Wilson loops in all the $\tilde{B}_i$'s, the only contributing assignments of the members of K to various SU(N) subgroups of $(\mathrm{SU}(N))^M$, that contribute to the "Lemma term" defined by R, are those which satisfy the requirement, that for each member i of R, all the W_j's, $j \in i$, are assigned to the *same* SU(N) subgroup of $(\mathrm{SU}(N))^M$. This means that, if for each assignment of all the W_j's, $j \in K$, to various SU(N) subgroups of $(\mathrm{SU}(N))^M$, we define the corresponding partition P of K to be the unique partition of K such that members i and j of K are members of the *same* member of P if and only if they are assigned to the *same* SU(N) subgroup of $(\mathrm{SU}(N))^M$, then the only such assignments that contribute to the "Lemma term" defined by the partition R of K, are those whose corresponding partition P satisfies the requirement that, for each member i of R, i is a *subset* of some member j of P.

3.1.3 The derivative with respect to M, at $M = 1$, becomes a derivative with respect to g^2

Furthermore, if all the W_j in the $(\mathrm{SU}(N))^M$ correlation function $\left[\prod_{j\in i} W_j\right]_{(\mathrm{SU}(N))^M}$ *are* assigned to the same $\mathrm{SU}(N)$ subgroup of $(\mathrm{SU}(N))^M$, then this $(\mathrm{SU}(N))^M$ correlation function is equal to the corresponding $\mathrm{SU}(N)$ correlation function $\left[\prod_{j\in i} W_j\right]_{\mathrm{SU}(N)}$, and it immediately follows from this that, as noted earlier, (in the discussion between equations (2.6) and (2.7)), the entire dependence of any term on its assignments of the members of K, to the various $\mathrm{SU}(N)$ subgroups of $(\mathrm{SU}(N))^M$, is through the corresponding partition P, and that, furthermore, when we sum over all those assignments that given the same P, the entire dependence on M is given, for this term, by the simple factor

$$M(M-1)(M-2)\dots(M-p+1) = \frac{M!}{(M-p)!}$$

(where $p = \mathbf{N}(P)$ is the number of parts of the partition P), times a factor $-\frac{(M-1)}{M}$ or $\frac{1}{M}$ for each 6-path, (where the choice depends on P and on the attachment points of the ends of that 6-path), times a similar factor for each Type-2 quartic vertex, as discussed above.

It follows that we may differentiate with respect to M and then set M equal to 1. We call the resulting equations the *group-variation equations for the* $\mathrm{SU}(N)$ *groups.*

Now

$$\frac{d}{dM}\left((NM)^{2-n-2r} f_r(W_1,\dots,W_n,g^2M)\right)\Big|_{M=1} =$$

$$= (2-n-2r)N^{2-n-2r} f_r(W_1,\dots,W_n,g^2) + N^{2-n-2r} g^2 \frac{d}{dg^2} f_r(W_1,\dots,W_n,g^2) \quad (3.8)$$

hence we immediately see, on substituting the expansions (2.7) into the left hand sides, that the group-variation equations for the $\mathrm{SU}(N)$ groups express the derivatives with respect to g^2 of the expansion coefficients in (1.55), in terms of those expansion coefficients themselves.

3.1.4 The Group-Variation Equation for a coefficient in the $\frac{1}{N}$ expansion of a vacuum expectation value or correlation function, involves only expansion coefficients of the same, or lower, non-vanishing order in the $\frac{1}{N}$ expansions

We shall now determine which of the f_s's in (1.55) can occur in the right-hand side of the group-variation equation for an expansion coefficient $f_r(W_1, \ldots, W_n, g^2)$ in (1.55).

For any right-hand side term of the type described above, we define l to be the total number of new paths minus the total number of action vertices, v to be the total number of 6-paths plus the total number of Type-2 quartic action vertices, k to be the total number of SU(N) fundamental representation Wilson loops, and s to be the total number of connected components of the decoration $\tilde{W}_1, \ldots, \tilde{W}_s$ of the original n Wilson loops $W_1, \ldots, W_n$. We also define, for each individual connected component $\tilde{W}_i$ of that decoration, l_i to be the total number of new paths minus the total number of action vertices of that connected component, v_i to be the total number of 6-paths plus the total number of Type-2 quartic action vertices of that connected component, k_i to be the total number of SU(N) fundamental representation Wilson loops of that connected component, and n_i to be the total number of the *original* n Wilson loops that form part of that connected component. Thus $l = \sum l_i$, $v = \sum v_i$, $k = \sum k_i$, and $n = \sum n_i$, where the sums all run from $i = 1$ to s.

One-loop vacuum bubbles are assigned $l = 0$ We note that there is no need to consider the "6-path" one-loop vacuum bubble, since it is completely independent of the $(\mathrm{SU}(N))^M$ gauge fields $A_{\mu a}$, and therefore cancels out.

Now every allowed decoration can be built up from the original n closed paths which define the n SU(NM) Wilson loops $W_1, \ldots, W_n$, plus one one-loop 45-path vacuum bubble for each $\tilde{W}_i$ that contains *none* of the original n Wilson loops, (and hence has $n_i = 0$, or in other words, is a vacuum bubble), by repeated applicatons of an "add a path" operation, similar to what we used for SU(N) Feynman diagrams. Each end of the path that is "added" at each stage may be either on a previously existing new path, or at a previously existing triple action vertex, or on an original path. We may build up the "core" decoration first, (that is, the decoration with all its 6-paths removed), and then add the 6-paths, (all of which have both their ends on the core decoration). Each added path increases the *total* number of new paths be either 3, 2, or 1, depending on where the ends of the added path go, and corresponding to these cases, the added

path increases the *total* number of action vertices by 2, 1, or 0, respectively, thus every "add a path" operation increases l by 1, and indeed l is equal to the total number of times the "add a path" operation has been performed. And furthermore, each l_i is equal to the number of "add a path" operations that have gone into the building of that $\tilde{W}_i$, where we note that a $\tilde{W}_i$, which by definition is a connected component of the full decoration, may, at early stages of the building process, have had more than one connected component.

We define one additional building operation, that consists of pinching together two previously existing 45-paths to form a Type-2 quartic action vertex, or pinching together two different points on a single previously existing 45-path, again to form a quartic action vertex. This operation increases the total number of new paths by 2, and increases the total number of action vertices by 1, hence again corresponds to increasing l by 1.

At each stage of the building of a valid decoration, a valid decoration is obtained, except that in some cases some of our selection rules, in particular the selection rule that it must not be possible to increase the number of connected components of a core decoration, by the removal of just one 45-path, may be disobeyed. But that doesn't matter, since our purpose is to derive, for each $\tilde{W}_i$, an inequality between k_i, l_i, v_i, and n_i, and this inequality is true even for those cases where these selection rules are disobeyed. We now find, by induction on l and on the individual l_i's, that for every $\tilde{W}_i$, the following inequality is satisfied:

$$k_i \leq 2 + l_i + v_i - n_i \tag{3.9}$$

and this immediately implies, on summing over i, that:

$$k \leq 2s + l + v - n \tag{3.10}$$

Indeed, (3.9) and (3.10) are satisfied by a single original closed path, which has $k = 1$, $l = v = 0$, $n = 1$, and a single one-loop 45-path vacuum bubble, which has $k = 2$, $l = v = 0$, $n = 0$, and also by any number of original closed paths and one-loop 45-path vacuum bubbles. And when a 45-path is added to a core decoration, k *increases* by 1 if both ends of the added 45-path "break into" the *same* previously existing SU(N) Wilson loop, and k *decreases* by 1 if each end of the added 45-path "breaks into" a *different* previously existing SU(N) Wilson loop. Furthermore, if the added 45-path has the effect of decreasing the number of connected components of the

core decoration by 1, then each of its ends is on a different previously existing connected component, hence each of its ends must "break into" a *different* previously existing Wilson loop, hence it results in k *decreasing* by 1. Hence (3.9) and (3.10) are preserved when the induction step consists of adding a 45-path to a core decoration. And when two previously existing 45-paths are pinched together to form a Type-2 quartic vertex, or two different points on a single previously existing 45-path are pinched together to form a Type-2 quartic vertex, k is unaltered, l and v each increase by 1, and s is either unaltered or else decreases by 1,(since this operation may pinch together two 45-paths, that are on different connected components of the previously existing core decoration), hence (3.9) and (3.10) are also preserved when the induction stip consists of forming a Type-2 quartic vertex by pinching togther two different points on the previously existing core diagram. And finally, adding a 6-path leaves k unaltered, increases l by 1, increases v by 1, and either leaves s unaltered or else decreases s by 1, (since each end of the added 6-path may be on a different connected component of the previously existing decoration), hence (3.9) and (3.10) are also preserved when the induction stip consists of adding a 6-path.

We note, as an immediate corollary to this proof, that if there is any 6-path or Type-2 quartic vertex whose removal, (in the case of a 6-path), or "unpinching", (in the case of a Type-2 quartic vertex), does *not* result in an increase of the number of connected components of the decoration, then the right-hand side of (3.10) may be replaced by:

$$2s + l + v - n - 2 \tag{3.11}$$

(Any decoration may be built up in the sequence: first build the core decoration with all its Type-2 quartic vertices "unpinched", then do the "pinchings" to form the Type-2 quartic vertices, then add the 6-paths.)

We note here that there is in fact an additional selection rule, namely that no decoration that contributes to the right-hand side of any of our $\mathrm{SU}(NM)/(\mathrm{SU}(N))^M$ group-changing equations or our $\mathrm{SU}(N)$ group-variation equations, can be such that its number of connected components can be increased by "unpinching" a single Type-2 quartic action vertex. This is because, as observed just before equation (3.7), the various terms associated with any Type-2 quartic action vertex at which four gauge-field paths meet, always exactly cancel out, so that we may assume that two of the paths that meet at any Type-2 quartic action vertex are Fadeev-Popov paths, and that that vertex arises from the $\psi_A f_{AaB} A_{\mu B} \phi_C f_{CaD} A_{\mu D}$ term in (1.23). And it immediately

follows from the form of this term, that there is *one* Fadeev-Popov path ending at that vertex at each "end" of the "source channel" of that vertex, i.e. at each end of the channel of that vertex through which *no* path-ordered phase factors pass. This means that when a Type-2 quartic action vertex is formed by pinching together two previously existing 45-paths, half of each of those two previously existing paths must become a Fadeev-Popov path, (the building process does not in general preserve the separate identities of gauge-field paths and Fadeev-Popov paths), and, in particular, if a Type-2 quartic action vertex is such that "unpinching" it results in increasing the number of connected ocmponents of the decoration, there would have to be one Fadeev-Popov path ending at that quartic action vertex in each of the two separate connected components that are produced by unpinching that Type-2 quartic action vertex. But Fadeev-Popov paths only occur in closed loops of Fadeev-Popov paths, hence the situation just described is impossible, since there is no way the two Fadeev-Popov paths ending at that "key" Type-2 quartic action vertex could form a closed loop of Fadeev-Popov paths. We note, however, that we *do* have to allow such "key" Type-2 quartic action vertices at intermediate stages in the process of building-up a decoration. (We do not distinguish gauge-field paths from Fadeev-Popov paths until the "building" of the decoration has been completed.)

We now continue with the determination of which of the f_s's in (1.55) can occur in the right-hand side of the group-variation equation for an expansion coefficient $f_r(W_1, \ldots, W_n, g^2)$ in (1.55), with the definitions of l, l_i, v, v_i, k, k_i, n, n_i, and s, and the sets K and S, as already introduced. Let us consider a term generated by our Lemma where we have r correlation-function factors, corresponding to the partition R of K, so that $r = \mathbf{N}(R)$. Now as we have already noted, (between equations (3.7) and (3.8)), the requirement that R be such that K is $(R \cup S)$-connected, implies that:

$$r \leq k + 1 - s \tag{3.12}$$

For each j, $1 \leq j \leq r$, let m_j be the number of members of the j^{th} member of R. We note that:

$$m_1 + m_2 + \ldots + m_r = k = \mathbf{N}(K) \tag{3.13}$$

We substitute in the expansion (1.55) for each of these correlation-function factors. Let us consider the term where, for the j^{th} correlation-function factor, $1 \leq j \leq r$, we take the b_j^{th} term in the expansion (1.55), or in other words, the term:

$$N^{2-m_j-2b_j} f_{b_j}(W_1, \ldots, W_{m_j}, g^2) \tag{3.14}$$

where $W_1, \ldots, W_{m_j}$ here are some of the SU(N) fundamental representation Wilson loops occurring in this decoration term, (specifically, those that are members of the j^{th} member of R), *not* the original n Wilson loops.

Then the total power of N, for this term, coming from the correlation-function factors, is:

$$\begin{aligned}(2-m_1-2b_1)+(2-m_2-2b_2)+\ldots+(2-m_r-2b_r) \qquad &= \\ = \qquad 2r-k-2(b_1+b_2+\ldots+b_r) \\ \leq \qquad 2(k+1-s)-k-2(b_1+b_2+\ldots+b_r) \\ = \qquad k+2-2s-2(b_1+b_2+\ldots+b_4) \\ \leq 2+l+v-n-2(b_1+b_2+\ldots+b_r) \qquad &(3.15)\end{aligned}$$

where at the last step here we used (3.10).

Now the factor $\frac{N}{4g^2}$ outside the action produces a factor $\frac{1}{N}$ for each new path and a factor N for each action vertex, hence an overall factor N^{-1}, and there is also, as already noted, an additional, explicit factor of $\frac{1}{N}$ associated with each 6-path, and with each Type-2 quartic action vertex, (coming from the explicit factors of $\frac{1}{N}$ in the appropriate terms in (1.59) and (2.6)), which gives an additional factor of N^{-v}.

Hence the *total* power of N for this term is:

$$\leq \qquad 2-n-2(b_1+b_2+\ldots+b_r) \qquad (3.16)$$

where we note that the inequality arises from the use of the upper bounds (3.10) and (3.12) on k and r.

Now $f_a(W_1, \ldots, W_n, g^2)$ in the left-hand side, (where $W_1, \ldots, W_n$ here are the n original Wilson loops), corresponds to a power of N of:

$$2-n-2a \qquad (3.17)$$

(see (3.8), with r in (3.8) replaced by a).

Hence, equating coefficients of powers of N in the original group-variation equations, we see that the term defined above can only contribute to the group-variation equation for $f_a(W_1, \ldots, W_n, g^2)$, if:

$$b_1+b_2+\ldots+b_r \leq a \qquad (3.18)$$

But all the b_j's are ≥ 0, hence *an f_b can only contribute to the right-hand side of the group-variation equation for f_a if $b \leq a$.*

3.1.5 The Group-Variation Equations are complete

Now the group-variation equations are *complete*. Indeed, we may directly recover the Feynman diagram expansions of the f_a's in powers of g^2 by developing the sums over paths in powers of g, (as we show how to do later in this paper, and as corresponds to developing the original background-field propagators in Section 2 in powers of g rather than as sums over paths), using the boundary condition that the one-Wilson-loop correlation function, (i.e. the one-Wilson-loop vacuum expectation value), is equal to 1 at $g^2 = 0$, and that all the other correlation functions are equal to 0 at $g^2 = 0$. (This procedure doesn't directly give the Feynman diagrams in one of the usual linear covariant gauges, but the usual Feynman diagrams can be recovered by the use of "propagator gauge invariance" identities, similar to those used by Mills [34], as we will show in detail in our next paper.)

The group-variation equations have, of course, to be renormalized, and also to be re-written to take account of the fact that we must divide a short-distance factor out of each Wilson loop. We indicate briefly how to do this later in this paper, and will give the full details in our next paper.

The important point for now is that, due to the result just obtained, we may first solve the equations for the f_0's, (which give the solution of the glueball sector of large-N_c QCD), *then* solve the equations for the f_1's, (using the f_0's already calculated), and so on.

3.1.6 Use of the renormalization group to express the derivative with respect to g^2

We note here that, once the group-variation equations have been renormalized, we will be able to express the derivative $\frac{d}{dg^2} f_a(W_1, \ldots, W_n, g^2)$ that occurs in (3.8), in terms of a derivative with respect to an overall linear scale factor L of the sizes and separations of the Wilson loops $W_1, \ldots, W_n$, by means of the renormalization group equation [35]:

$$\left(L\frac{\partial}{\partial L} + \beta(g)\frac{\partial}{\partial g} \right) f_a(W_1, \ldots, W_n, g^2) = 0 \tag{3.19}$$

(In practice there will be additional terms in this equation due to the short-distance factor which we have to divide out.)

When (3.19) is used, the boundary conditions at $g^2 = 0$ quoted above will be

replaced by the behaviours of the f_a's as $L \to 0$, as determined by renomalization-group-improved perturbation theory.

3.2 Simplifications at leading non-vanishing order in $\frac{1}{N}$

We shall devote the remainder of this paper to the analysis of the group-variation equations for the f_0's, which, as we shall soon see, have the advantage, over the direct sum of Feynman diagrams, that their solution manifestly has the correct qualitative behaviour, namely, the Wilson area law for the one-Wilson-loop vacuum expectation value, and massive glueball saturation for the multi-Wilson-loop correlation functions, and that, moreover, this correct qualitative behaviour is obtained for the very simplest approximation to the solution, and is maintained in all higher approximations.

Let l, l_i, v, v_i, k, k_i, n, n_i, and s, and the sets K, S, and R, be defined, for each right-hand side term, as in the preceding discussions.

We begin by noting some further simplifications that occur in the right-hand sides of the group-variation equations for the f_0's.

We first note that for $a = 0$, the inequality in (3.18) becomes an equality, (and of course, all the b_j are equal to 0), which immediately implies that kk and r must be equal to their upper limits as given by (3.10) and (3.12) respectively.

Now the requirement that k be equal to its upper limit as given by (3.10) implies, firstly, that at each step of building up the core of an allowed decoration $\tilde{W}_1 \dots \tilde{W}_s$ from the initial n Wilson loops, plus as many one-loop 45-path vacuum bubbles as there are vacuum bubbles among the $\tilde{W}_i$'s, the added 45-path must satisfy the requirement that both its ends "break into" the *same* previously existing SU(N) fundamental representation Wilson loop, except at those steps where this is *impossible*, due to the added 45-path having each end in a different previously existing connected component, (and thus reducing the total number of connected components by 1). And secondly, as immediately follows from the discussion accompanying formula (3.11), there must be *no* 6-pat or Type-2 quartic vertex whose removal, (in the case of a 6-path), or "unpinching", (in the case of a Type-2 quartic vertex), does *not* result in an increase of the number of connected components of the decoration. But by the discussion following formula (3.11), there can be *no* such Type-2 quartic vertex in the full decoration, hence

there must be *no Type-2 quartic vertices at all.*

Now r is to be equal to its maximum possible value, as given by (3.12), and this implies, in particular, that if i is any member of S, and j is any member of R, then $i \cap j$ has at most one member. For if $i \cap j$ had two or more members then we could remove all but one of the members of $i \cap j$ from j, and for each member u of $i \cap j$ that we remove from j, add the one-member set $\{u\}$ to R, (thus increasing $r = \mathbf{N}(R)$), and K would remain $(R \cup S)$-connected, contradicting the assumption that r has its maximum possible value consistent with K being $(R \cup S)$-connected.

(We note that the upper bound (3.12) on r follows from the following consideration: for given K and S, we form R by a sequence of steps starting from an "initial" R which is defined to be the set of all the one-members subsets of K. Thus the "initial" R has $k = \mathbf{N}(K)$ members, and K has $s = \mathbf{N}(S)$ $(R \cup S)$-connected components. We transform this "initial" R to the given R by a sequence of steps, each of which consists of replacing two members of "R at that stage", by their union. Thus each step results in reducing $r = \mathbf{N}(R)$ by 1. Now each step may or may not reduce the number of $(R \cup S)$-connected components of K by 1, but certainly no step reduces the number of $(R \cup S)$-connected components of K by *more* than 1, since each step consists of replacing just two members of R by their union, and no member of R intersects more than one $(R \cup S)$-connected component of K, hence no two members of R intersect more than two $(R \cup S)$-connected components of K. Hence to reach an R such that the number of $(R \cup S)$-connected components of K has been reduced from s to 1, there must be at least $s - 1$ of these steps.)

We now return for a moment to the group-changing equations for $(\mathrm{SU}(N))^M$ and $\mathrm{SU}(NM)$, and consider a 6-pat, with its ends "attached", (in the usual sense - see e.g. the discussion before and after (2.20)), to specific $\mathrm{SU}(N)$ fundamental representation Wilson loops. We consider the sum over the assignments of which of the M $\mathrm{SU}(N)$ subgroups of $(\mathrm{SU}(N))^M$ the $\mathrm{SU}(N)$ fundamental representation Wilson loops to which that 6-path attaches, are "in", (or in other words, that their gauge-fields belong to). Let us *suppose* that we can sum over the M possibilities completely independently at each end of that 6-path, (so that we assume, in particular, that each end of that 6-path attaches to a *different* $\mathrm{SU}(N)$ fundamental representation Wilson loop). We then see immediately , from the 6-term in (2.6), that from the M possibilities where we have the *same* $\mathrm{SU}(N)$ subgroup of $(\mathrm{SU}(N))^M$ at each end of the 6-path, we get a contribution that includes, from the overall factor M and from the $\left(-\delta_{JQ} + \frac{1}{M}\right)$ factor in the 6-

term, the factor $M\left(-\frac{(M-1)}{M}\right) = -(M-1)$, while from the $M(M-1)$ possibilities where we have a *different* SU(N) subgroup of $(\mathrm{SU}(N))^M$ at each end of the 6-path, we get the same contribution, except that from the overall factor $M(M-1)$ and from the $\left(-\delta_{JQ} + \frac{1}{M}\right)$ factor in the 6-term, we now get the factor $M(M-1)\left(\frac{1}{M}\right) = (M-1)$. Thus *the total is zero*: we have yet another "selection rule". We note that this holds for general M, and is thus also true for the group-*variation* equations for the SU(N) groups.

We did not mention this selection rule before, because the circumstances under which it can be applied are extremely restricted: we must be able to sum completely independently over the "which SU(N)" assignments of the two SU(N) fundamental representation Wilson loops to which that 6-path attaches, and every other factor in the integrand of the term under consideration, must be completely independent of whether or not those two SU(N) fundamental representation Wilson loops are in the same part, or in different parts of the partition P of K that is defined, as described between (3.7) and (3.8), for each assignment of the SU(N) fundamental representation Wilson loops to various SU(N) subgroups of $(\mathrm{SU}(N))^M$.

We now return to the analysis of the terms that occur in the right-hand sides of the group-variation equations for the f_0's.

We have already seen that there can be no Type-2 quartic vertices at all, and that any 6-path that occurs must be such that its removal results in an increase in the number of connected components of the decoration. And we have also seen that the fact that for each right-hand side term, $r = \mathbf{N}(R)$ is equal to its maximum possible value, as given by (3.12), implies that for each right-hand side term, if i is any member of S and j is any member of R, then $i \cap j$ has at most one member. Let us consider a right-hand side term that has a 6-path whose removal results in an increase in the number of connected components of that decoration. Then each end of that 6-path certainly attaches to a *different* SU(N) fundamental representation Wilson loop, indeed the SU(N) fundamental representation Wilson loops to which the ends of that 6-path attach are in different connected components of the decoration that remains when that 6-path is removed. And furthermore, by the result just mentioned, these two SU(N) fundamental representation Wilson loops are members of *different* members of R, since they are members of the same member of S. Furthermore, it also follows from the fact that r is equal to its maximum possible value, that if we define $\tilde{S}$ to be the partition of K that is obtained from S by removing the member of S that

contains the SU(N) fundamental representation Wilson loops of the connected component of our decoration which contains the 6-path we are considering, and replacing it by the two (nonempty) sets of SU(N) fundamental representation Wilson loops that correspond to the two connected components into which that connected component of our decoration separates when we remove that 6-path, (so that the members of $\tilde{S}$ are in one-to-one correspondence with the connected components of the decoration we obtain by removing that 6-path, and the members of each member of $\tilde{S}$ are the SU(N) fundamental representation Wilson loops in the corresponding connected component of that decoration), then K is *not* $(R \cup \tilde{S})$-connected. For $\mathbf{N}(\tilde{S}) = \mathbf{N}(S)+1 = s+1$, hence by the same proof as before, if K was $(R \cup \tilde{S})$-connected, then R could have at most $k+1-s-1 = k-s$ members, which contradicts the assumption that R has $k+1-s$ members. Hence there exists a partition of K into two nonempty parts, say U and V, such that *no* member of $(R \cup \tilde{S})$ intersects both U and V. Now if i is the member of S that we removed in forming $\tilde{S}$ from S, and u and v are the two members of $\tilde{S}$ into which i has "split", then u is a subset of one member of $\{U, V\}$, and v is a subset of the other member of $\{U, V\}$, (for u and v are members of $\tilde{S}$, hence neither u nor v intersects both U and V, while if both u and v were subsets of the *same* member of $\{U, V\}$, then the fact that K is $(R \cup S)$-connected, hence that some member of $(R \cup S)$ intersects both U and V, and the fact that $i = u \cup v$ is a subset of *one* member of $\{U, V\}$, hence does *not* intersect both members of $\{U, V\}$, would imply that some member of $(R \cup \tilde{S})$ intersects both U and V, contrary to assumption). Suppose for definiteness that u is a subset of U and v is a subset of V. Then one end of our 6-path attaches to a member of u, hence a member of U, while the other end of our 6-path attaches to a member of v, hence a member of V. Now the fact that no member of R intersects both U and V implies that the product of the correlation functions generated by our Lemma for this term, completely factorizes into a factor that depends only on the members of U, and a factor that depends only on the members of V. And furthermore, the fact that no member of $\tilde{S}$ intersects both U and V implies that apart from our 6-path itself, the integrand for this term completely factors into a factor associated with U, and a factor associated with V. Hence the conditions for the applicability of the selection rule which we derived just above, are satisfied: apart from the $\left(\delta_{JQ} + \frac{1}{M}\right)$ factor in the 6-term in (2.6), the integrand for this term is completely independent of whether the two SU(N) fundamental representation Wilson loops to which the ends of our 6-path attach, are in the same, or different, SU(N) subgroups of $(\mathrm{SU}(N))^M$, so that we sum

over all possibilities, the result is zero.

It immediately follows from this thata no decoration that includes a 6-path whose removal results in increasing the number of connected components of that decoration, gives any contribution, hence, combining this with the previous results, we see that *there are no 6-paths and no Type-2 quartic vertices at all in the right-hand sides of the group-variation equations for the f_0's.*

Hence every contributing decoration term can be obtained from one-loop 45-path vacuum bubbles and the n original Wilson loops by repeated applications of the simple "add a path" procedure, with every added path being a 45-path, so that all the new paths are 45-paths.

We can now give a simple topological interpretation of each term, as defined above, in the right-hand side of the group-changing equation for any f_0. Let the integer $n \geq 1$ denote, as usual, the number of Wilson loop arguments of the f_0 whose group-changing equation we are considering, and let the integers l, l_i, k, k_i, n_i, and s, and the sets K, S, and R, have their usual meanings of the term under consideration, so that the suffix i runs over the connected components of the decoration under consideratiaon of the initial n Wilson loops, l is the total number of new paths minus the total number of action vertices, l_i is the corresponding quantity for the i^{th} connected component of the decoration, K is the set of all the SU(N) fundamental representation Wilson loops into which the SU(N) fundamental representation path-ordered phase factors in the 45-paths and the original paths are connected by the specific routeings through the action vertices defined by the term under consideration, $k = \mathbf{N}(K)$, S is the partition of K whose parts are in one-to-one correspondence with the s connected components of the decoration, such that the k_i members of the i^{th} part of S are precisely those members of K that belong to the i^{th} connected component fo the decoration, n_i is the number of the n original Wilson loops that form part of the i^{th} connected component of the decoration, and R is the partition of K which defines, in accordance with the Lemma, the product of correlation functions of members of K, to which the particular "Lemma term" that we are considering, corresponds, (so that R satisfies the requirement that K is $(R \cup S)$-connected). The integers v and v_i defined previously are of course zero in the present situation, by the results just derived.

We first note that the requirement that k be equal to its maximum possible value, as given by (3.10), implies that if we "fill" each member of K, (i.e. each SU(N) fundamental representation Wilson loop), with an oriented topological 2-disk, then for each

individual connected component of the decoration, the k_i such oriented topological 2-disks associated with that connected component join up to form an oriented topological 2-sphere with n_i holes, where the boundaries of the holes are the n_i original Wilson loops that form part of that connected component, and the boundaries of the holes are all oriented the same way. The "joining up" process consists of "sewing together", along each 45-path, the two oriented topological 2-disks whose boundaries meet along that 45-path. (We recall that one of our selection rules is that no member of K passes in both directions along any 45-path.) The reasoning is exactly as given for Feynman diagrams before (1.54), with "closed loop of Kronecker deltas" now replaced by "SU(N) fundamental representation Wilson loop", or in other words, "member of K".

We note that the oriented topological 2-sphere with n_i holes is equivalent to the oriented topological 2-disk with $n_i - 1$ holes, so that (1.54) shows a possible example with $n_i = 3$.

3.2.1 Topological representation of the right-hand side terms

We shall now construct a topological representation of our right-hand side term, in which each member T_j of R, $1 \leq j \leq r$, where $r = \mathbf{N}(R)$, is represented by an oriented topological 2-sphere with $t_j = \mathbf{N}(T_j)$ holes, with the boundaries of the holes being the t_j members of K that are members of T_j, and all the boundaries of the holes being oriented the same way, i.e. such that their orientations as defined by their path-orderings are consistent with the orientation of that topological 2-sphere. (We recall that each member T_j of R corresponds to a factor, in the term under consideration, equal to the correlation function of the members of T_j.)

We now see that the topological construction that we have already given, where we filled *every* member of K with an oriented topological 2-disk,(or in other words, an oriented topological 2-sphere with one hole), irrespective of whether that member of K was as member of a one-member member of R, or a member of a member T_j of R with two or more members, corresponds to carrying out the above construction for the "initial" R that is defined to be the set of all the one-member subsets of K, as we described in the proof given above, (some paragraphs after (3.19)), of the upper limit (3.12) on r. We now proceed to "correct" this "initial" topological construction into a topological construction that exactly corresponds to the "true" R by the above rule, in a series of steps that exactly correspond to the steps that we used, in the above-

mentioned proof of the upper limit on r, to transform the "initial" R into the "true" R.

We note first that in this proof given above, (some paragraphs after (3.19)), of the upper limit (3.12) on r, each step in the process of transforming the "initial" R, (i.e. the set of all the one-member subsets of K), to the "true" R, consists of replacing two members of the "R at that stage" by their union, and thus reducing the number of members of R by 1. Now such a step reduces by 1 the number of $(R \cup S)$-connected components of K if the two members of the "R at that stage", which we replace by their union, were subsets of two *different* $(R \cup S)$-connected components of K, while it has no effect on the number of $(R \cup S)$-connected components of K if the two members of the "R at that stage" which we replace by their union, were subsets of the *same* $(R \cup S)$-connected component of K. Now the upper bound (3.12) on r comes from noting that to reduce the number of $(R \cup S)$-connected components of K from its initial value of s, to 1, there must be *at least* $s - 1$ such steps, and we now see that to have $r = \mathbf{N}(R)$ equal to its maximum possible value of $k + 1 - s$, as given by (3.12), (which, as we have already seen, is a necessary requirement for our term to contribute to the right-hand side of the group-variation equation for an f_0), we must be able to construct the "true" R from the "initial" R in *exactly* $s - 1$ such steps, which immediately implies that at each step, the two members of the "R at that stage" which we replace by their union, must be subsets of *different* $(R \cup S)$-connected components of K, (so that the number of $(R \cup S)$-connected components of K decreases by 1 at each step.)

Now the topological equivalent of replacing two members of the "R at that stage" by their union is given by the following consideration: at the first step, since all the members of the "initial" R are one-member subsets of K, corresponding to 2-spheres with one hole each, with the boundary of each being a member of K, we are to form the union of two one-member subsets of K. We cut a small hole in each of the "2-spheres with one hole" whose boundaries are the two relevant members of K, and join those two "2-spheres with one hole" to one another by a long tube, (which is itself a topological 2-sphere with two holes), sewing each end of the tube to one of the "small" holes which we have just made in each of those two "2-spheres with one hole", in such a way that the orientations of the two "2-spheres with one hole" are consistent with one another, and also with the orientation of the tube, or cylinder. We see immediately that the result of this operation is an oriented topological 2-sphere with two holes, exactly as required by our rule for a two-member member of R. Now it is a general property of

oriented topological 2-spheres with holes [36] that if we have two oriented topological 2-spheres, one with u holes and one with v holes, and we make a small additional hole in each sphere, and join the two spheres together by a long cylinder, sewing each end of the cylinder to one of the small additional holes we just made in each of the spheres, in such a way that the orientations of the two spheres are consistent with one another and also consistent with the orientation of the cylinder, then the result is an oriented topological 2-sphere with $u + v$ holes. (It is not, of course, necessary for the additional holes to be "small", or for the cylinder to be "long": it simply helps with the visualization of the process. In fact the cylinder or tube could be dispensed with, and the two spheres sewn directly to ne another at the boundaries of the additional holes, but we retain it to help with the visualization.) Thus every step of the process by which we transform the "initial R" into the "true R", (each step consisting of replacing two members of the "R at that stage" by their union), can be reproduced exactly in the topological picture, by joining the appropriate two "oriented topological 2-spheres with holes" (which correspond to the two members of the "R at that stage" whose union we are forming), into a new oriented topological 2-sphere with holes, whose set of holes is the union of the sets of holes of the original two 2-spheres.

Now what happens to the s oriented topological 2-spheres with holes, the i^{th} having n_i holes, that correspond to the s connected components of the decoration, (and which we formed as described above, by filling *every* member of K by an oriented topological 2-sphere with one hole), when we carry out these operations step by step, corresponding exactly to the transformation of the "initial R", in exactly $s - 1$ steps, to the "true R"? We see immediately that because, (in order to reach the "true R" in *exactly* $s-1$ steps, as is necessary for $r = \mathbf{N}(R)$ to be equal to the upper limit as given by (3.12)), we must, as just shown, at every step, form the union of two members of the "R at that stage" that are subsets of two *different* $(R \cup S)$-connected components of K, every step reduces the number of connected components of our "topological model" of our term by 1, and every step consists of joining two oriented topological 2-spheres with holes, into a single oriented topological 2-sphere with holes, by the tube procedure, where this last conclusion follows by induction on the number of steps so far performed. In fact at each step, the connected components of our topological model at that stage, exactly correspond to the $(R \cup S)$-connected components of K at that stage. *Thus at the end of the procedure, (i.e. when we have reached the "true" R, after having performed* $s-1$ *steps), our topological model for this term consists of exactly one oriented topological*

2-sphere, with n holes, with the boundaries of the holes being the initial n Wilson loops, each oriented, (in the sense of its path-ordering), consistently with the orientation of this one oriented topological 2-sphere.

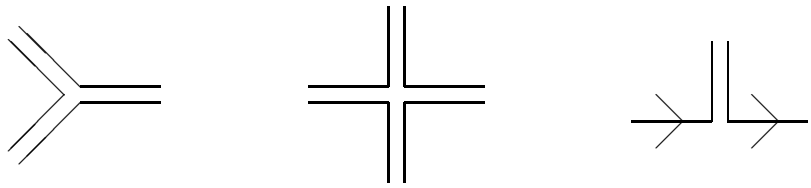
We note that to reach this conclusion that the topological model of our term is an oriented topological 2-*sphere*, as opposed to a surface of higher genus, (i.e. with handles), it is crucial that, at each step of the above procedure, the two members of the "R at that stage" whose union we form, are subsets of two *different* $(R \cup S)$-connected components of K, (and thus members of two *different* connected components of the topological model at that stage), for if we were to make the "two small holes", at any step, in a *single* connected component of the topological model at that stage, the result of sewing the ends of the cylinder to the two small holes would be a surface of higher genus, i.e. with a handle.

Now as we have already remarked, an oriented topological 2-sphere with n holes is equivalent to an oriented topological 2-disk with $n-1$ holes, (since of course we assume $n \geq 1$), so we see that we may represent the topological model of our term by a diagram where we choose one of our original n Wilson loops as the outer boundary of a 2-disk, and draw the other $n-1$ Wilson loops inside this outer boundary, (but *not* within one another), and draw the 45-paths of the decoration on the oriented 2-disk, with $n-1$ holes, thus formed. We immediately see that we can draw all the 45-paths without any crossings of their SU(N) fundamental representation path-ordered phase factors, and respecting our rule "drive on the left", provided that the *outermost* boundary of the disk is oriented *anti-clockwise*, (i.e. has its path-ordering arrows pointing *anti-clockwise*), while each of the $n-1$ "inner boundaries" are oriented *clockwise*. (This is precisely what is required for all n boundaries of this disk, when it is stretched to look like a 2-sphere with n holes, to be oriented the same way.) We can now immediately read off this diagram what correlation function factors we have in this term: a simply-connected "window", (or "region"), of this diagram corresponds to the vacuum expectation value of the single SU(N) fundamental representation Wilson loop which forms its perimeter, a doubly-connected "window" corresponds to the correlation function of the two SU(N) fundamental representation Wilson loops which form its perimeter, and so on.

(For historical reasons, we note that "window-weighted" path integrals have previously been considered in the context of large-N_c QCD by Migdal and Makeenko [37]. However our equations have absolutely no connection at all with theirs.)

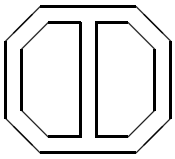
And furthermore, we can go the other way: given the oriented 2-disk with $n-1$ holes,

and its boundaries oriented as described, and any planar diagram of 45-paths drawn on this disk, with *no* crossings of SU(N) fundamental representation path-ordered phase factors, and the only vertices being:



(3.20)

(where the single line with arrows represents part of one of the boundaries of the disk), we obtain a diagram which precisely corresponds to the topological model of one of our terms, provided the number of connected components of the *drawing*, (i.e. of the *decoration*, *not* the topological model), cannot be increased by deleting just one 45-path, (this corresponds t one of our selection rules). The diagram *can*, of course, have "islands", such as:



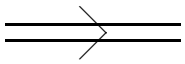
(3.21)

In such a case the window surrounding the islands corresponds, by our general rule, to the correlation function of the SU(N) fundamental representation Wilson loops that form the outer boundaries of the islands and the outer perimeter of that window.

It is appropriate to note here that planar diagrams don't usually have symmetry factors associated with specific subdiagrams. This is because if a subdiagram of a Feynman diagram or decoration is left invariant by a certain group of permutations of its lines and vertices, there is usually precisely the right number of "leading large-N" routeings of its Kronecker deltas or path-ordered phase factors through its vertices, to cancel its symmetry factor. However some planar diagrams contributing to Wilson loop vacuum expectation values, and similarly some of our present diagrams, have symmetry factors corresponding to "rotational" symmetries of the entire diagram. These symmetries, and the corresponding symmetry factors, vanish if we arbitrarily assign one point on each Wilson loop as the "base point" of that loop, past which no vertex on that loop may go. But if we do that, we immediately get a larger number of the less symmetric diagrams, which would be equivalent to one another but for inequivalent assignments of which adjacent pair of their vertices on the loop the base point goes between. We will display our diagrams assuming that such base points are *not* introduced, so that in some cases we will need to introduce symmetry factors.

Furthermore, in our diagrams we sometimes need symmetry factors associated with individual "islands", due to rotational symmetries of those islands. For example, the island (3.21) needs the symmetry factor $\frac{1}{2}$, (provided none of its paths are Fadeev-Popov paths). The corresponding vacuum bubble, (with three gauge-field paths), has the symmetry factor $\frac{1}{6}$, but there are three possible choices of which one of the three SU(N) fundamental representation Wilson loops of that vacuum bubble, to make into the outside of the island, (i.e. to include in the same member of R as some member of K from another connected component of the decoration).

It is now appropriate to introduce a notation to distinguish Fadeev-Popov 45-paths from gauge-field 45-paths, and we do this by putting a big arrow on each Fadeev-Popov 45-path:



(3.22)

We may consider the arrow, on a Fadeev-Popov propagator, as pointing from the ψ_A field to the ϕ_A field, hence all these arrows should point in the same cyclic direction around any closed loop of Fadeev-Popov paths.

For the remainder of this paper, any 45-path without such a big arrow, is to be interpreted as a gauge-field path.

We now return for a moment to the group-changing equations for $(\mathrm{SU}(N))^M$ and SU(NM), in order to determine the M-dependence of the terms corresponding to our diagrams, so that we can differentiate with respect to M, then set $M = 1$. We have two fundamental rules:

1. If two SU(N) fundamental representation Wilson loops share any 45-path, they must be in *different* SU(N) subgroups of $(\mathrm{SU}(N))^M$. (This follows from the $(1 - \delta JK)$ factor in the 45-term in (2.6).)

2. All the SU(N) fundamental representation Wilson loops involved in any correlation function must be in the *same* SU(N) subgroup of $(\mathrm{SU}(N))^M$. (This follows from the fact that the $(\mathrm{SU}(N))^M$ correlation function of a set of SU(N) fundamental representation Wilson loops vanishes unless all the SU(N) fundamental representation Wilson loops in that set are in the *same* SU(N) subgroup of $(\mathrm{SU}(N))^M$. And if they *are* all in the same SU(N) subgroup of $(\mathrm{SU}(N))^M$, then that $(\mathrm{SU}(N))^M$ correlation function is equal to the corresponding SU(N) correlation function.)

With reference to (1) here, we note that our "planar diagram" rules, namely that we use only the vertices (3.20), and that there must be no crossings of SU(N) fundamental representation path-ordered phase factors, and our rule that it must not be possible to increase the number of connected components of the decoration by deleting a single 45-path, together imply, as required, that no SU(N) fundamental representation Wilson loop can pass in both directions along any 45-path. This follows by induction on l, exactly as for the corresponding result for planar Feynman diagrams, (see the discussion between (1.51) and (1.52)).

The implication of (2) for our diagrams is simple: For each window of our diagram, irrespective of whether that window is simply-connected or multiply-connected, *all* the SU(N) fundamental representation Wilson loops that form the boundary of that window, must be in the *same* SU(N) subgroup of $(\mathrm{SU}(N))^M$.

3.2.2 Chromatic Polynomials

Thus we see, if we disregard for a moment any possible complications associated with the vertices, that we have an analogue of the famous "map-colouring" problem. Any allowed assignment of the members of K, i.e. the SU(N) fundamental representation Wilson loops, ot the various SU(N) subgroups of $(\mathrm{SU}(N))^M$, consists of an assignment, consistent with (1) and (2) above, of an integer in $\{1, 2, \ldots, M\}$ to each SU(N) fundamental representation Wilson loop. We have to calculate the M-dependence of the set of all such allowed assignments, defined for all M as the lowest-degree polynomial in M that gives the correct result for all integers $M \geq 0$, (since this is what is given by the general form of the M-dependence as discussed, e.g., before equation (3.8), and is also consistent with the "simplest sum of powers" by which the N-dependence of the Feynman diagrams is generalized to all N), then we have to calculate the derivative with respect to M at $M = 1$.

Instead of M *numbers* we may think of M "*colours*". We then see immediately that (2) implies that instead of colouring the *lines*, i.e. the SU(N) fundamental representation Wilson loops, we may colour the windows, (since (2) says that all the lines that form the boundary of any window, are to be coloured the same colour). Then (1) is the classic map-colouring requirement that "countries" that share a border must be coloured in different colours. The polynomial $\mathbf{C}(M)$, defined as above to be the lowest-degree polynomial that gives the correct number of distinct colourings, with M

available colours, for all integer $M \geq 0$, is sometimes called the "chromatic polynomial". We note that some of our "countries" are of course allowed to be non-simply connected, i.e. to enclose or surround other countries, or one or more of the $n-1$ "interior" holes, or both. However we do *not* have any "disconnected" countries, i.e. countries with two or more connected components, as is sometimes allowed in the map-colouring problem.

Now, are there any complications coming from the vertices?

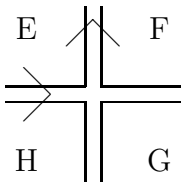
We note first that the triple vertices certainly cause no problems. For only two windows meet at an original-path vertex, and those two windows are forced by their common 45-path to be different colours, and similarly only three windows meet at an action vertex, and they are again forced by their common 45-paths to have three different colours.

However four windows meet at a quartic vertex, and the constraint (1) allows four different "partitions" (into like-coloured subsets) of the set of these four windows: all four may be different colours, one opposite pair may be like-coloured, the other opposite pair being unlike-coloured, (two different choices here, corresponding to which is the like-coloured opposite pair), or both opposite pairs may be like-coloured. However it immediately follows from the discussion in the paragraph beginning "Certain simplifications occur", (after(3.6)), that if all four paths ending at a quartic vertex are gauge-field paths, then when we add the contributions of the $f_{aBC}f_{aEF}$ and the $f_{ABC}f_{AEF}$ terms in (1.31), which give respectively the $-\delta_{jq}\delta_{kp}$ term in (1.59), and the whole of (1.60), the result is simply the $-\delta_{JQ}\delta_{jq}\delta_{KP}\delta_{kp}$ term in (1.62), which is completely independent of which of the M colours each of the four SU(N) path-ordered phase factors passing through the vertex is in. In fact, the $f_{aBC}f_{aEF}$ term in (1.31) gives the contribution of all the "colour assignments" where both the phase-factor lines passing through the "source channel" of that vertex, (i.e. the 2/2 channel which has one structure constant at each "end"), have the *same* colour, while the $f_{ABC}f_{AEF}$ term in (1.31) gives the contribution of all the "colour assignments" where the two phase-factor lines passing through the "source channel" of the vertex are assigned two *different* colours. Thus we see that a quartic vertex at which four gauge-field paths end gives *no complications at all*: the simple map-colouring rules given above, which take no account at all of whether opposite pairs of windows at quartic vertices are like or unlike coloured, are exactly correct.

However if two of the paths that end at a quartic vertex are Fadeev-Popov paths,

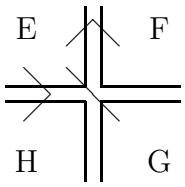
then the only possible source of that vertex is the term $\psi_A f_{AaB} A_{\mu B} \phi_C f_{CaD} A_{\mu D}$ in (1.23), which is a "123-term", in the sense that the suffix a is summed only over the 1's, 2's, and 3's in (1.58). Thus this Type-1 quartic vertex only gets the Type-1 contribution from (1.59), namely the $-\delta_{JA}\delta_{KA}\delta_{PA}\delta_{QA}\delta_{jq}\delta_{kp}$ term in (1.59), summed over A from 1 to M, to give $-\delta_{JQ}\delta_{PK}\delta_{JK}\delta_{jq}\delta_{pk}$, (where the summation convention is of course *not* applied to J or K). Thus this vertex only contributes "colour assignments" where the two phase-factor lines passing through it "source channel" are the *same* colour. This means that at such a quartic vertex in our diagrams, the opposite pair of windows at that vertex which lie on either *side* of the "source channel" of that vertex, are to be coloured the *same* colour. In other words, for the purposes of window colourings, it is just as though that vertex has been sliced into two across its "source channel", so that the opposite pair of windows, which lie on either side of the "source channel" of that vertex, are "merged" into a single window.

Now the structure of the $\psi_A f_{AaB} A_{\mu B} \phi_C f_{CaD} A_{\mu D}$ term in (1.23) shows that such a vertex has one Fadeev-Popov "leg" at each end of its "source channel". Thus if the two Fadeev-Popov "legs" of the vertex share a phase-factor line, or in other words, if the two Fadeev-Popov "legs" of that vertex are "neighbours" in the planar diagram, for example:



(3.23)

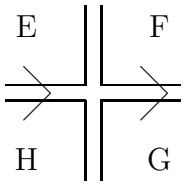
then the "source channel" of the vertex is fixed uniquely. In example (3.23) the windows at the *ends* of the source channel are F and H, and the windows at the *sides* of the source channel are E and G. Therefore, for purposes of calculating the chromatic polynomial $\mathbf{C}(M)$ for one of our diagrams containing the vertex (3.23), this vertex is to be "sliced into two" by a cut from window E to window G, so that windows E and G are merged into a single window:



(3.24)

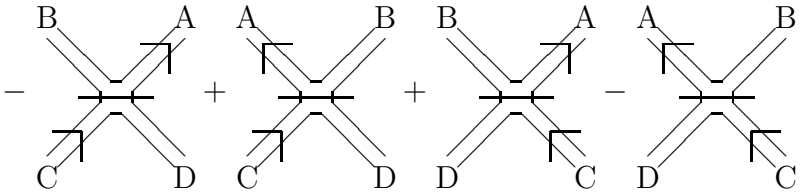
However if the two Fadeev-Popov "legs" of a quartic vertex at which two Fadeev-Popov paths end, do *not* share any phase-factor line, or in other words, if the two

Fadeev-Popov "legs" of the vertex are "opposite legs" in the planar diagram, for example:



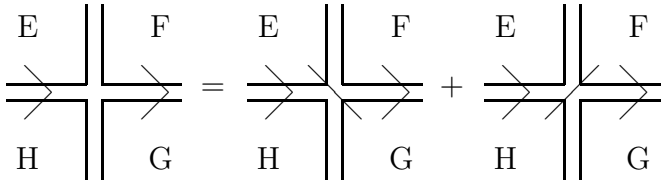
(3.25)

then the "source channel" of the vertex is *not* fixed uniquely: its ends could be windows E and G, or windows F and H. Now use of (2.1) shows that $f_{AaB}f_{CaD}$, with the conventions of our diagrams, has the structure:



(3.26)

The first and last terms here show that whenever the vertex (3.25) occurs in our diagrams, we are, for the purposes of calculating the chromatic polynomial $\mathbf{C}(M)$ of the diagram, to sum over the two possible identifications of the "source channel" of the vertex. Thus we adopt the diagrammatic identity:



(3.27)

(We may confirm that this is correct by symmetrizing $f_{AaB}f_{CaD}$ under $B \rightleftharpoons D$.)

For practical purposes, we may consider the "slice" through such a vertex as simply connecting together the two windows which are to be coloured in the same colour. (The other opposite pair of windows, i.e. the pair *separated* by the "slice", may be either like-coloured or unlike-coloured.)

Thus a vertex like (3.25), shown without a "slice", is to be interpreted by (3.27). We don't usually display the "slice" for a vertex like (3.23), since the only place it can go is as shown in (3.24).

Thus, in summary, the chromatic polynomial $\mathbf{C}(M)$ is to be calculated, for each of our diagrams, as the lowest-degree polynomial in M which gives correctly, for every integer $M \geq 0$, the number of distinct ways of colouring the windows of that diagram with M available colours, subject to the requirement that if two windows share a common "boundary", (i.e. a common 45-path), they are to be coloured in two *different*

colours, and subject to the special rules, as just explained, for windows that meet at a quartic vertex that has two Fadeev-Popov "legs".

We then differentiate the chromatic polynomial $\mathbf{C}(M)$ with respect to M, and evaluate the derivative $\frac{d}{dM}\mathbf{C}(M)$ at $M = 1$. This gives the coefficient with which that diagram occurs in the right-hand side of the group-variation equation for the appropriate f_0. (For the purposes of the present discussion, we consider the possible "symmetry factors", associated with "rotational symmetries" of the diagram, as discussed above, to be an "intrinsic" part of the mathematical expression corresponding to the diagram.)

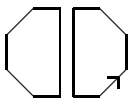
Chapter 4

Vanishing Of The Chromatic Polynomial Factor When There Is More Than One Island, Effective Mass For The 45-Paths From The Window Weights, and Minimal-Length Spanning Trees

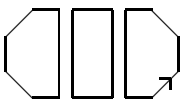
4.1 Examples of the diagrams, and their chromatic polynomials

We now consider some examples of our diagrams, and their associated chromatic polynomials $\mathbf{C}(M)$, first for $n = 1$:

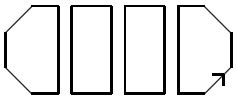
$$\mathbf{C}(M) = M \tag{4.1}$$



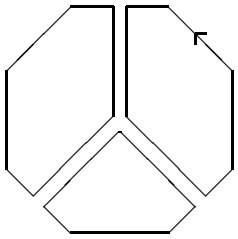
$$\mathbf{C}(M) = M(M-1) \tag{4.2}$$



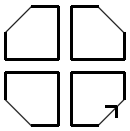
$$\mathbf{C}(M) = M(M-1)^2 \tag{4.3}$$



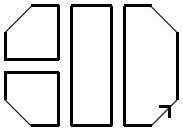
$$\mathbf{C}(M) = M(M-1)^3 \tag{4.4}$$



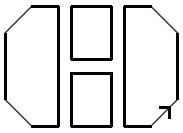
$$\mathbf{C}(M) = M(M-1)(M-2) \tag{4.5}$$



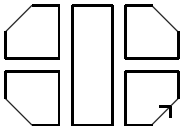
$$\mathbf{C}(M) = M(M-1)(M^2-3M+3) \tag{4.6}$$



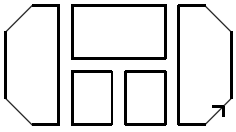
$$\mathbf{C}(M) = M(M-1)^2(M-2) \tag{4.7}$$



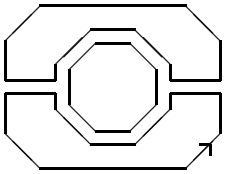
$$\mathbf{C}(M) = M(M-1)(M-2)^2 \tag{4.8}$$



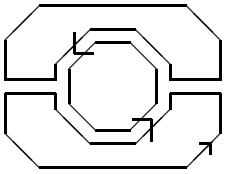
$$\mathbf{C}(M) = M(M-1)^2(M-2)^2 \tag{4.9}$$



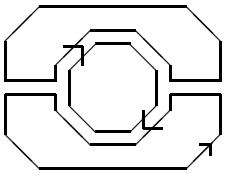
$$\mathbf{C}(M) = M(M-1)(M-2)^3 \tag{4.10}$$



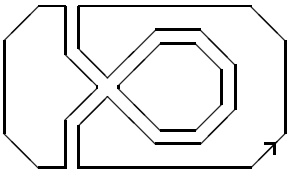
$$\mathbf{C}(M) = M(M-1)(M-2) \tag{4.11}$$



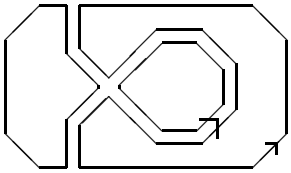
$$\mathbf{C}(M) = M(M-1)(M-2) \tag{4.12}$$



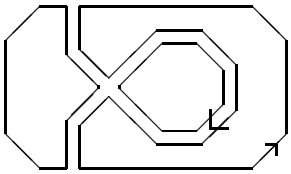
$$\mathbf{C}(M) = M(M-1)(M-2) \tag{4.13}$$



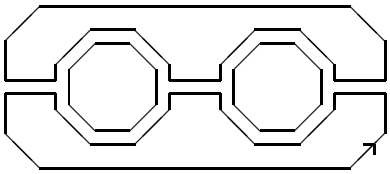
$$\mathbf{C}(M) = M(M-1)^2 \tag{4.14}$$



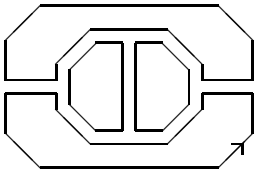
$$\mathbf{C}(M) = M(M-1) \tag{4.15}$$



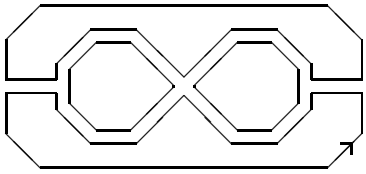
$$\mathbf{C}(M) = M(M-1) \tag{4.16}$$



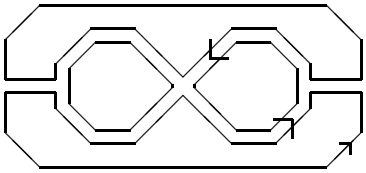
$$\mathbf{C}(M) = M(M-1)(M-1)^2 \tag{4.17}$$



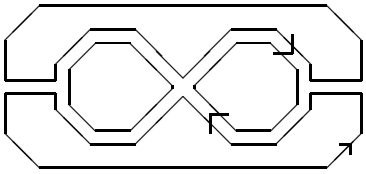
$$\mathbf{C}(M) = M(M-1)(M-2)(M-3) \tag{4.18}$$



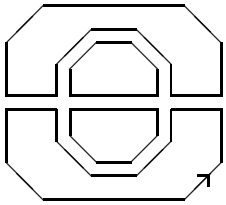
$$\mathbf{C}(M) = M(M-1)(M-2)^2 \tag{4.19}$$



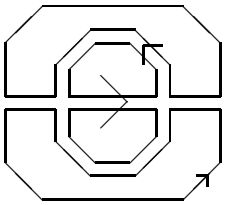
$$\mathbf{C}(M) = M(M-1)(M-2) \tag{4.20}$$



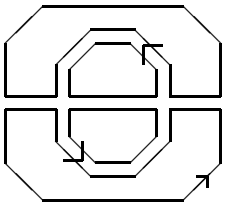
$$\mathbf{C}(M) = M(M-1)(M-2) \tag{4.21}$$



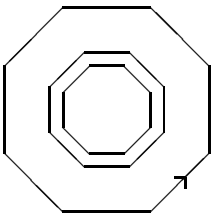
$$\mathbf{C}(M) = M(M-1)^2(M-2) \tag{4.22}$$



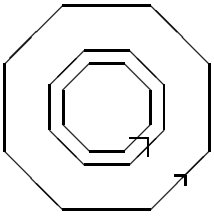
$$\mathbf{C}(M) = M(M-1)^2 \tag{4.23}$$



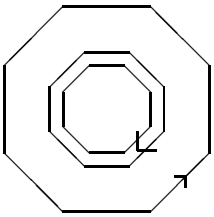
$$\mathbf{C}(M) = 2M^2(M-1) \tag{4.24}$$



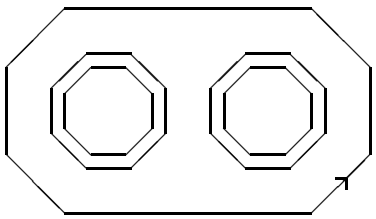
$$\mathbf{C}(M) = M(M-1) \tag{4.25}$$



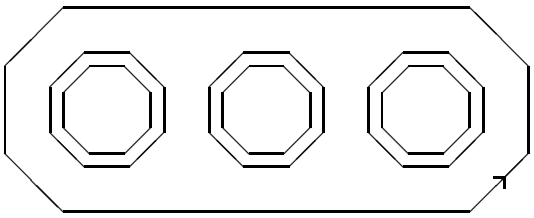
$$\mathbf{C}(M) = M(M-1) \tag{4.26}$$



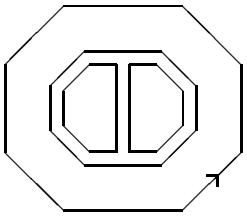
$$\mathbf{C}(M) = M(M-1) \tag{4.27}$$



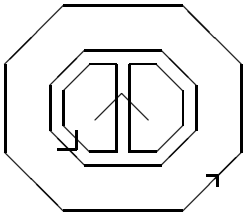
$$\mathbf{C}(M) = M(M-1)^2 \tag{4.28}$$



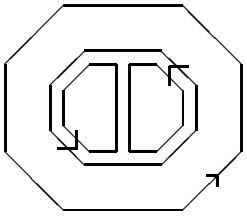
$$\mathbf{C}(M) = M(M-1)^3 \tag{4.29}$$



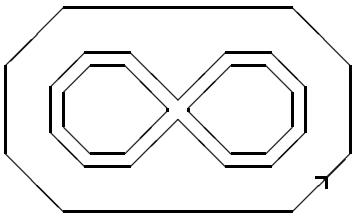
$$\mathbf{C}(M) = M(M-1)(M-2) \tag{4.30}$$



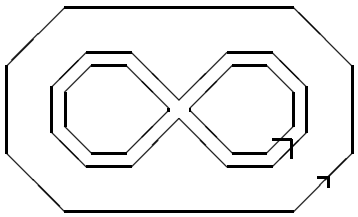
$$\mathbf{C}(M) = M(M-1)(M-2) \tag{4.31}$$



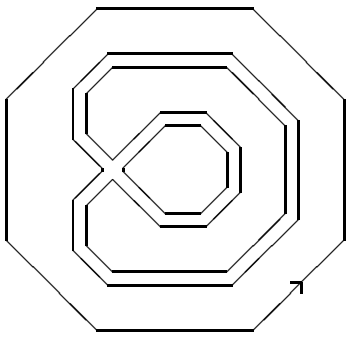
$$\mathbf{C}(M) = M(M-1)(M-2) \tag{4.32}$$



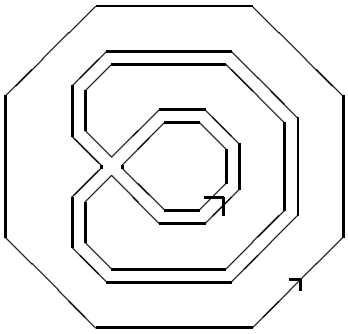
$$\mathbf{C}(M) = M(M-1)^2 \tag{4.33}$$



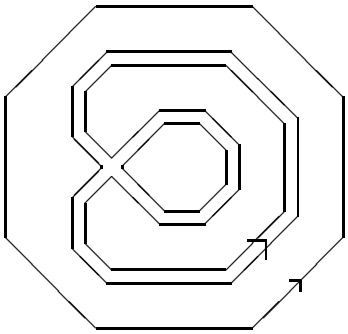
$$\mathbf{C}(M) = M(M-1) \tag{4.34}$$



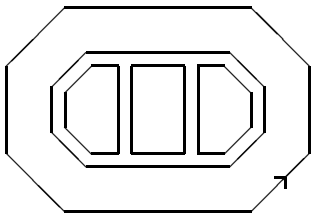
$$\mathbf{C}(M) = M(M-1)^2 \tag{4.35}$$



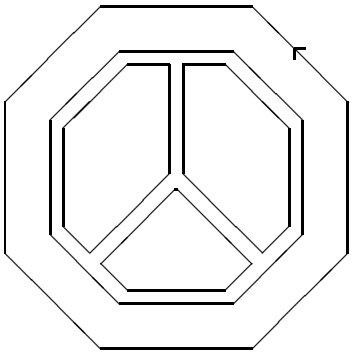
$$\mathbf{C}(M) = M(M-1) \tag{4.36}$$



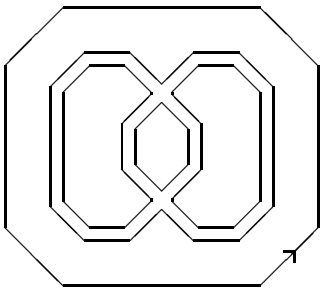
$$\mathbf{C}(M) = M(M-1) \tag{4.37}$$



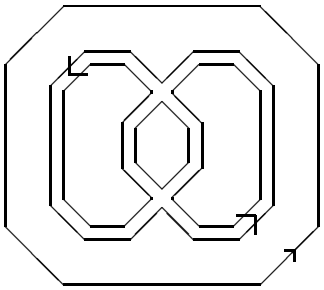
$$\mathbf{C}(M) = M(M-1)(M-2)^2 \tag{4.38}$$



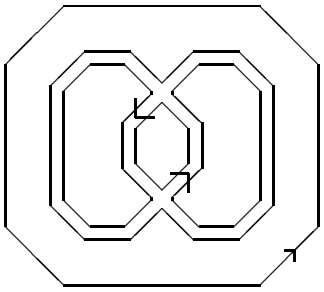
$$\mathbf{C}(M) = M(M-1)(M-2)(M-3) \tag{4.39}$$



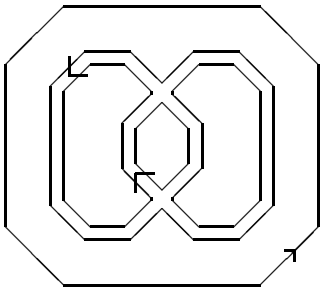
$$\mathbf{C}(M) = M(M-1)(M^2-3M+3) \tag{4.40}$$



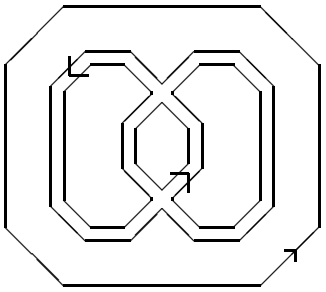
$$\mathbf{C}(M) = M(M-1)^2 \tag{4.41}$$



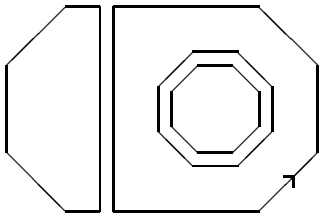
$$\mathbf{C}(M) = M(M-1)^2 \tag{4.42}$$



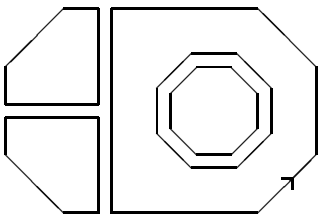
$$\mathbf{C}(M) = M(M-1)^2 \tag{4.43}$$



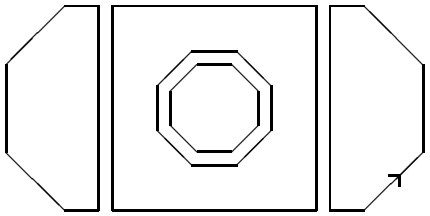
$$\mathbf{C}(M) = 2M^2(M-1) \tag{4.44}$$



$$\mathbf{C}(M) = M(M-1)^2 \tag{4.45}$$



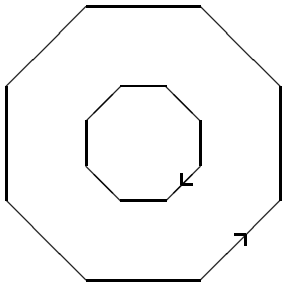
$$\mathbf{C}(M) = M(M-1)^2(M-2) \tag{4.46}$$



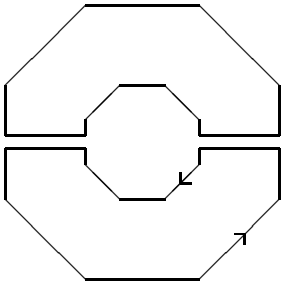
$$\mathbf{C}(M) = M(M-1)^3 \tag{4.47}$$

We note that the chromatic polynomial $\mathbf{C}(M)$ may be calculated for any of our diagrams as the sum, over all permitted partitions of the set of all the windows into like-coloured sets, of $M(M-1)\dots(M-p+1) = \frac{M!}{(M-p)!}$, where p is the number of parts of the partition. However it may frequently be calculated more quickly by other methods.

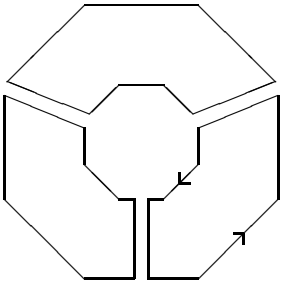
We next consider some examples for $n = 2$:



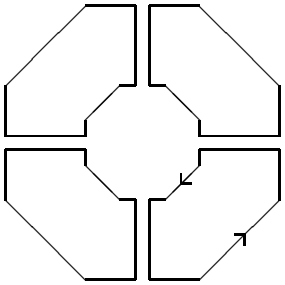
$$\mathbf{C}(M) = M \tag{4.48}$$



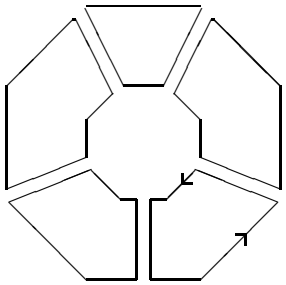
$$\mathbf{C}(M) = M(M-1) \tag{4.49}$$



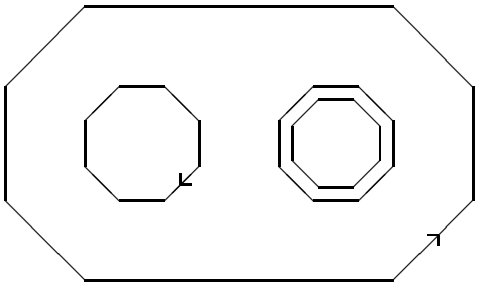
$$\mathbf{C}(M) = M(M-1)(M-2) \tag{4.50}$$



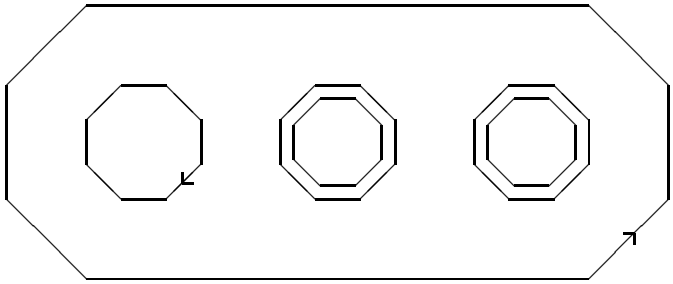
$$\mathbf{C}(M) = M(M-1)(M^2-3M+3) \tag{4.51}$$



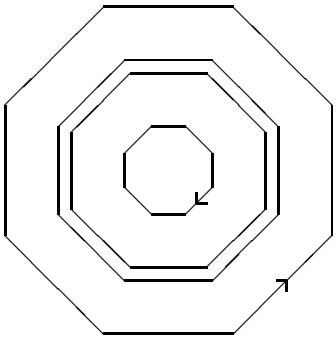
$$\mathbf{C}(M) = M(M-1)(M-2)(M^2-2M+2) \tag{4.52}$$



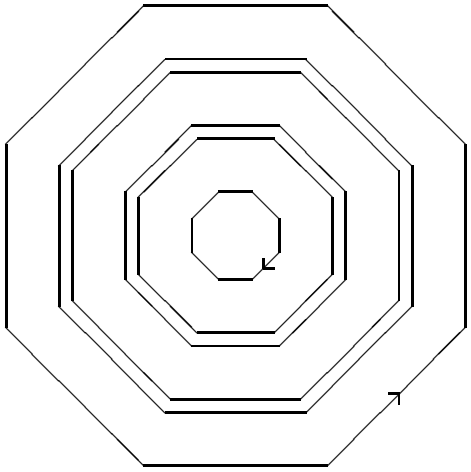
$$\mathbf{C}(M) = M(M-1) \tag{4.53}$$



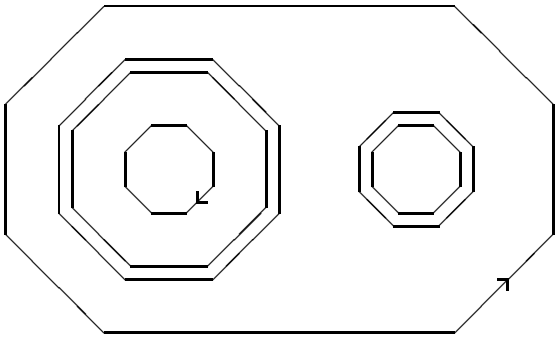
$$\mathbf{C}(M) = M(M-1)^2 \tag{4.54}$$



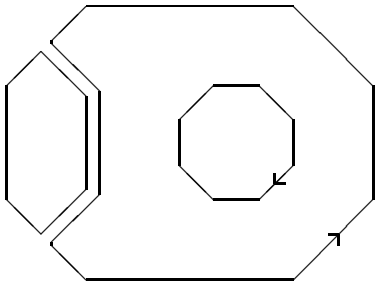
$$\mathbf{C}(M) = M(M-1) \tag{4.55}$$



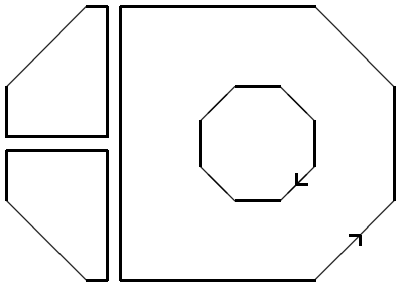
$$\mathbf{C}(M) = M(M-1)^2 \tag{4.56}$$



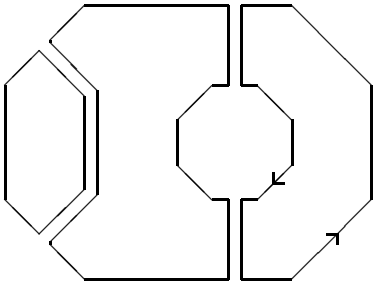
$$\mathbf{C}(M) = M(M-1)^2 \tag{4.57}$$



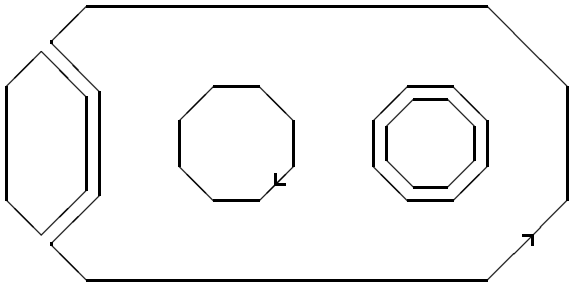
$$\mathbf{C}(M) = M(M-1) \tag{4.58}$$



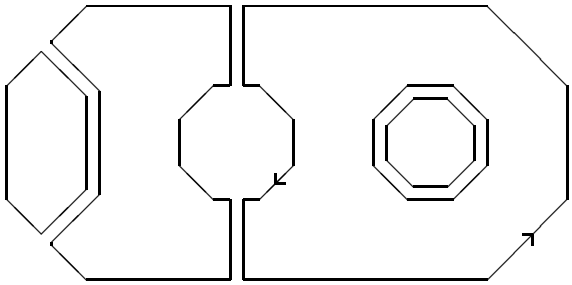
$$\mathbf{C}(M) = M(M-1)(M-2) \tag{4.59}$$



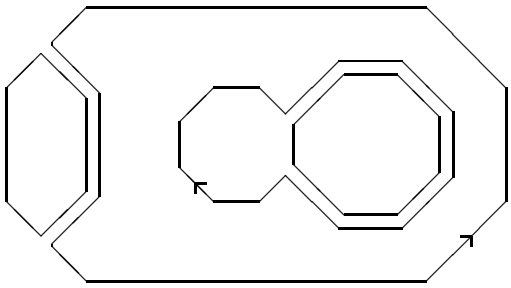
$$\mathbf{C}(M) = M(M-1)^2 \tag{4.60}$$



$$\mathbf{C}(M) = M(M-1)^2 \tag{4.61}$$

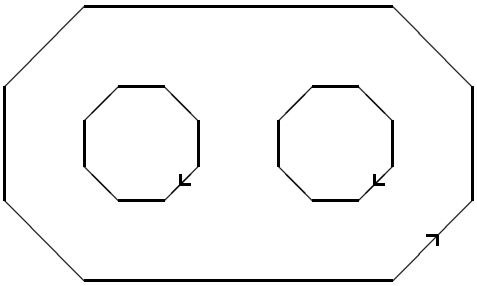


$$\mathbf{C}(M) = M(M-1)^3 \tag{4.62}$$

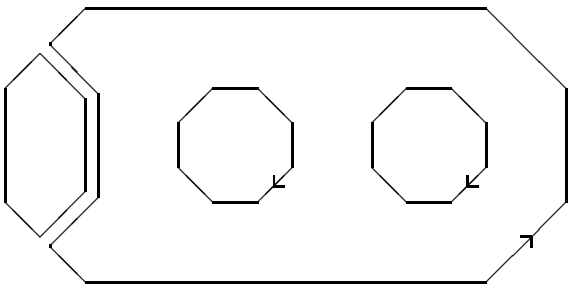


$$\mathbf{C}(M) = M(M-1)^2 \tag{4.63}$$

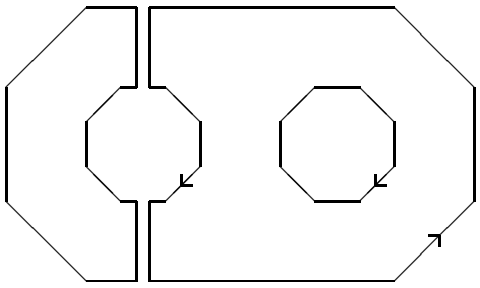
and some examples for $n = 3$:



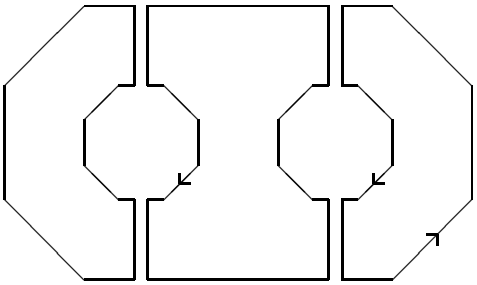
$$\mathbf{C}(M) = M \tag{4.64}$$



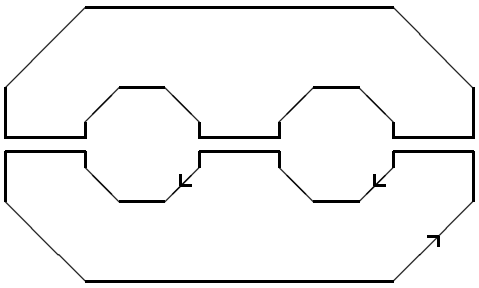
$$\mathbf{C}(M) = M(M-1) \tag{4.65}$$



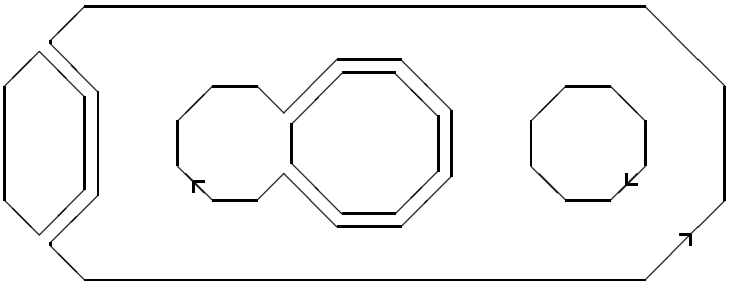
$$\mathbf{C}(M) = M(M-1) \tag{4.66}$$



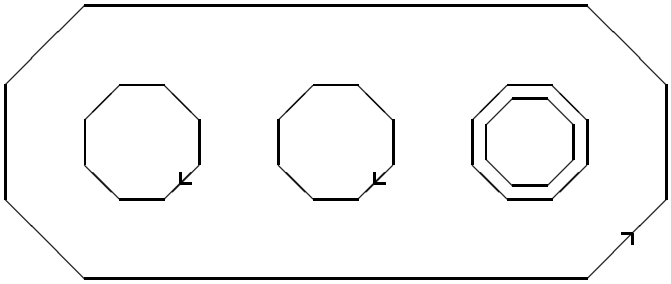
$$\mathbf{C}(M) = M(M-1)^2 \tag{4.67}$$



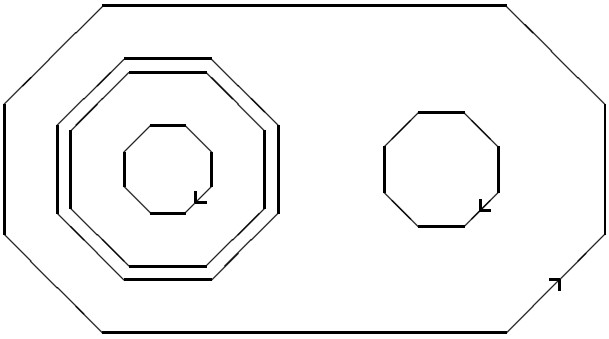
$$\mathbf{C}(M) = M(M-1) \tag{4.68}$$



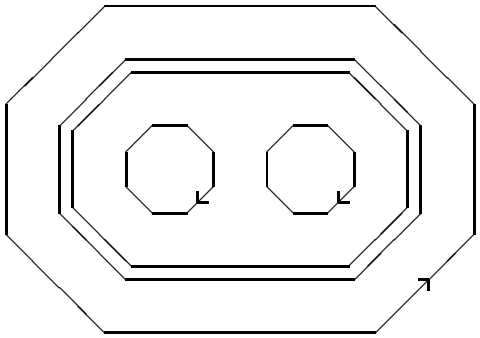
$$\mathbf{C}(M) = M(M-1)^2 \tag{4.69}$$



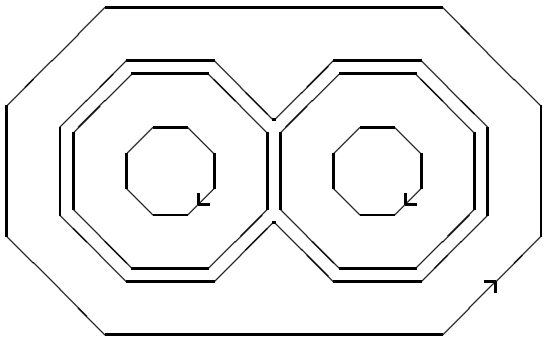
$$\mathbf{C}(M) = M(M-1) \tag{4.70}$$



$$\mathbf{C}(M) = M(M-1) \tag{4.71}$$



$$\mathbf{C}(M) = M(M-1) \tag{4.72}$$



$$\mathbf{C}(M) = M(M-1)(M-2) \tag{4.73}$$

4.2 Circumstances that guarantee that $\mathbf{C}(M)$ has at least two factors of $(M-1)$

We now observe, firstly, that for *every* diagram, apart from those, namely (4.1), (4.48), and (4.64), that have *no* 45-paths, $\mathbf{C}(M)$ includes at least one factor of $(M-1)$. This is indeed exactly what we expect, since it is impossible to colour a diagram that includes 45-paths with just one colour.

But we also observe that for a large number of the diagrams, $\mathbf{C}(M)$ includes two or more factors of $(M-1)$, so that $\frac{d}{dM}\mathbf{C}(M)\Big|_{M=1}$ is equal to zero, *so that that diagram makes no contribution at all to the right-hand side of the relevant group-variation equation.*

Indeed for *any* diagram, such that the set U of all the windows of that diagram, can be partitioned into three nonempty sets X, Y, and Z, such that X contains exactly one window, and no member of Y shares a 45-path with any member of Z, $\mathbf{C}(M)$ will have at least two factors of $(M-1)$. For $\mathbf{C}(M)$ in such a case factors into a factor M for the one window in X, and a factor associated with Y, and a factor associated with Z, and the factor associated with Y and the factor associated with Z *each* contain at

least on factor of $(M-1)$. (Indeed, $\mathbf{C}(M)$ for any such diagram is equal to $\frac{1}{M}$, times the chromatic polynomial for the diagram obtained from the given one by removing all the 45-paths that separate the windows in Y from one another and from the window in X, times the chromatic polynomial for the diagram obtained from the given one by removing all the 45-paths that separate the windows in Z from one another and from the window in X, and each of these last two chromatic polynomials includes at least one factor of $(M-1)$.)

Hence for any diagram, such that the set U of all the windows of that diagram, can be partitioned into three nonempty parts X, Y, and Z, such that X contains exactly one window, and no window in Y shares any 45-path with any window in Z, the coefficient $\frac{d}{dM}\mathbf{C}(M)\Big|_{M=1}$ vanishes, *so that that diagram makes no contribution at all to the right-hand side of the relevant group-variation equation.*

This is a crucial result of this paper, since it immediately eliminates vast classes of diagrams from the right-hand sides of the group-variation equations.

We now define, in the context of our diagrams, an "island" to be, precisely, a vacuum bubble. Thus an island is a connected component of a diagram, that does *not* involve any of the n original Wilson loops. In the above examples, the diagrams with islands are: (4.25) - (4.47) inclusive, (4.53)-(4.57) inclusive, (4.61), (4.62), and (4.70) - (4.73) inclusive. And among these, (4.29) has three islands, (4.28), (4.54), (4.56), and (4.57) each have two islands, and the remainer all have one island.

We now see that it follows immediately from the above results, that if a diagram has an island, then it must have *no* 45-paths other than those that form part of that island, if it is to give a nonvanishing contribution. Indeed, the chromatic polynomial of a diagram with an island, is equal to $\frac{1}{M}$, times the chromatic polynomial of the diagram obtained from the given diagram by removing all 45-paths that do *not* form part of the island, times the chromatic polynomial of the diagram obtained from the given diagram by removing the island, and if there are any 45-paths that do *not* form part of the island, then *both* these last two factors will include a factor $(M-1)$. *Hence we immediately conclude, in particular, that if a diagram is to give a nonvanishing contribution, it must contain at most one island, and, if it* does *have an island, it must have* no *45-paths that do* not *form part of that island.*

We call a diagram an "island diagram" if it has an island, and a "non-island diagram" if it has no island.

We next define, in the context of our diagrams, a "band" to be a connected compo-

nent of what remains of a diagram after we remove all the islands and all the "original paths", (i.e. all the paths that form wholes or parts of the n original Wilson loops). In the above examples, the diagrams with *one* band are (4.2), (4.5), (4.6), (4.8), (4.10) - (4.24) inclusive, (4.45), (4.46), (4.58), (4.59), (4.61), and (4.65), the diagrams with *two* bands are (4.3), (4.7), (4.9), (4.47), (4.49), (4.63), (4.66), and (4.69), the diagrams with *three* bands are (4.4), (4.50), (4.60), (4.62), and (4.68), the diagrams with *four* bands are (4.51) and (4.67), and the diagram with *five* bands is (4.52). We note that each band has at least two "free ends", (i.e. where a 45-path ends at an original path).

We now find another "selection rule": if a diagram has a band, all of whose free ends are on a *single* one of the n original closed paths, then that diagram must have *no* 45-paths, apart from those in that band, if it is to give a nonzero contribution. This implies, in particular, that a diagram that gives a nonvanishing contribution to the right-hadn side of the group-variation equation for $f_0(W_1, g^2)$, has at most one band.

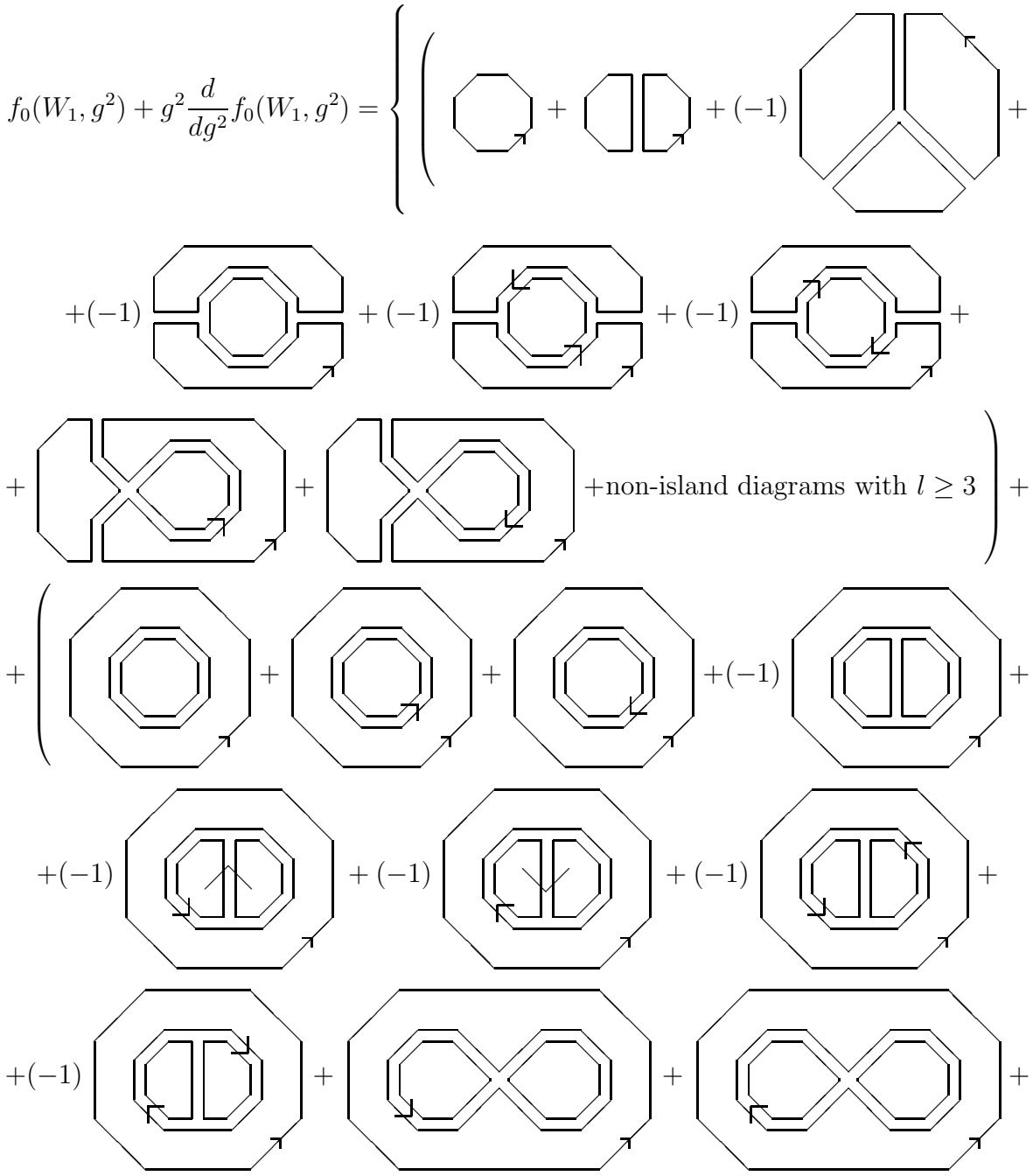
There is, however, for $n \geq 2$, no such limit on the number of bands, each of which has its free ends on two or more *different* original closed paths. Indeed, the generalization of examples (4.49) - (4.52) to the "b-spoked wheel", ($b \geq 2$), contributes, for $n = 2$, with the coefficient $(-1)^b$. (To see this, we pick one of the b windows, and classify the allowed colourings by the total number, $c \geq 1$, of the b windows that are coloured the same colour as the chosen window. The contribution to $\mathbf{C}(M)$ from colourings with a total of c windows coloured the same as the chosen window, is given by $M(M-1)^c(M-2)^{b-2c}$ times an integer coefficient that counts the number of ways of choosing the $(c-1)$ windows that are to be coloured the same as the chosen window. Only $c = 1$ contributes to the derivative at $M = 1$, and the number of ways in this case is 1.)

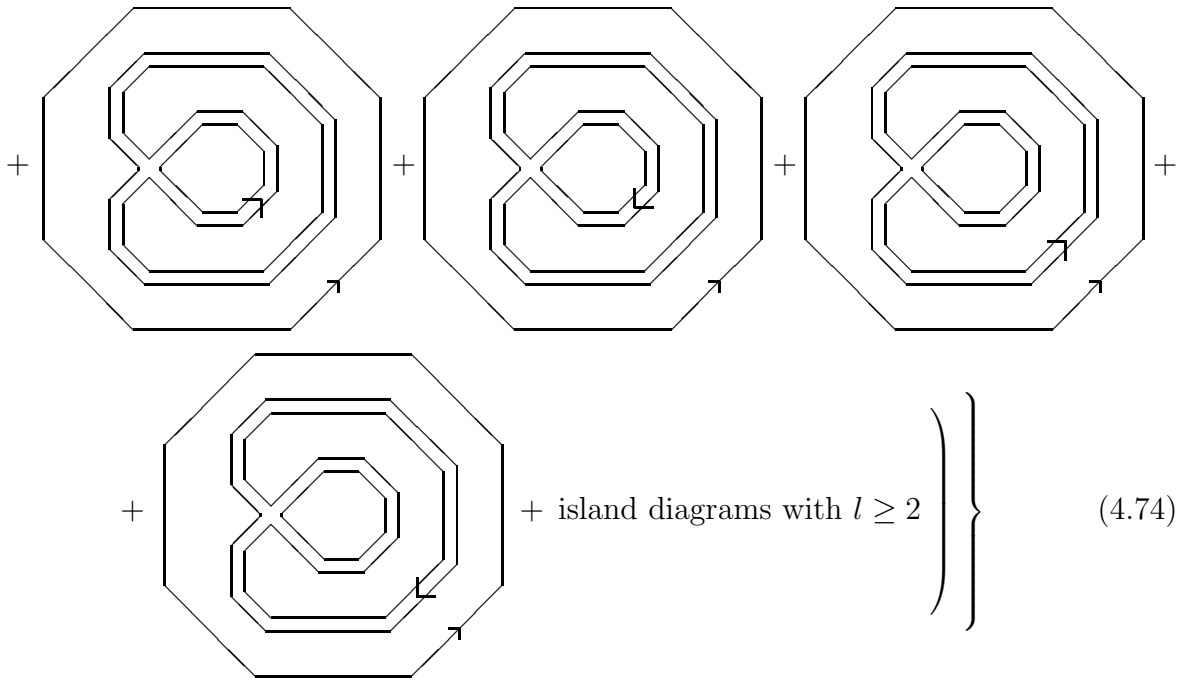
We note that, for each diagram that gives a nonvanishing contribution, apart from the diagrams that have *no* 45-paths, the chromatic polynomial $\mathbf{C}(M)$ includes precisely one factor of $(M-1)$. Thus evaluating the derivative at $M = 1$ reduces to removing the factor of $(M-1)$, and evaluating what remains at $M = 1$.

4.3 The Group-Variation Equation for the vacuum expectation value of one Wilson loop

The explicit form of the group-variation equation for $f_0(W_1, g^2)$ is:

$f_0(W_1, g^2) + g^2 \frac{d}{dg^2} f_0(W_1, g^2) =$

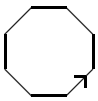




(4.74)

Here l denotes, as usual, the number of new paths, (i.e. 45-paths), minus the number of action vertices, and we continue to use the convention, as stated before (4.1), that where a diagram has a "symmetry factor" associated with a rotational symmetry, (for example, a factor $\frac{1}{2}$ for the diagram (4.2), and a factor $\frac{1}{3}$ for the diagram (4.5)), this factor is considered to be an intrinsic part of the mathematical expression corresponding to the diagram, and is thus not displayed explicitly.

The left-hand side of (4.74) is as given by (3.8), with $r = 0$ and $n = 1$. Now the first term in the right-hand side of (4.74), namely the diagram:



(4.75)

is simply the diagrammatic form of $f_0(W_1, g^2)$, so we could of course cancel this term between the two sides of (4.74). However, each side of equation (4.74) has a very simple form in terms of *Feynman* diagrams, as follows: each side of equation (4.74) is equal to the sum of all the leading large-N *Feynman* diagrams that contribute to the vacuum expectation value of the Wilson loop W_1, with each *Feynman* diagram being simply multiplied by its number of windows, (i.e. by its number of closed loops of Kronecker deltas).

In fact, for each $n \geq 1$, each side of the group-variation equation for $f_0(W_1, \ldots, W_n, g^2)$, when written with the left-hand side being given by (3.8), with the

appropriate n, and $r = 0$, (and of course, the overall power of N cancelled out), is equal to the sum of all the *Feynman* diagrams contributing to $f_0(W_1, \ldots, W_n, g^2)$, with each *Feynman* diagram being simply multiplied by its number of windows.

Indeed, as follows immediately from the discussion preceding (1.55), each Feynman diagram contributing to $f_0(W_1, \ldots, W_n, g^2)$ with t powers of g^2, has precisely $t + 2 - n$ windows, i.e. $t + 2 - n$ closed loops of Kronecker deltas. Thus each Feynman diagram with u windows that contributes to $f_0(W_1, \ldots, W_n, g^2)$, has precisely $u + n - 2$ powers of g^2. Hence $g^2 \frac{d}{dg^2}$ simply has the effect of multiplying each u-windowed Feynman diagram by a factor $(u + n - 2)$, which immediately gives the result stated.

We therefore choose the definitive form of each group-variation equation, (for purposes of referring to its "left-hand side" or its "right-hand side"), to be that where the left-hand side is given by the appropriate case of (3.8), (with the overall power of N cancelled out), in order to retain this simple interpretation, in terms of Feynman diagrams, for $r = 0$.

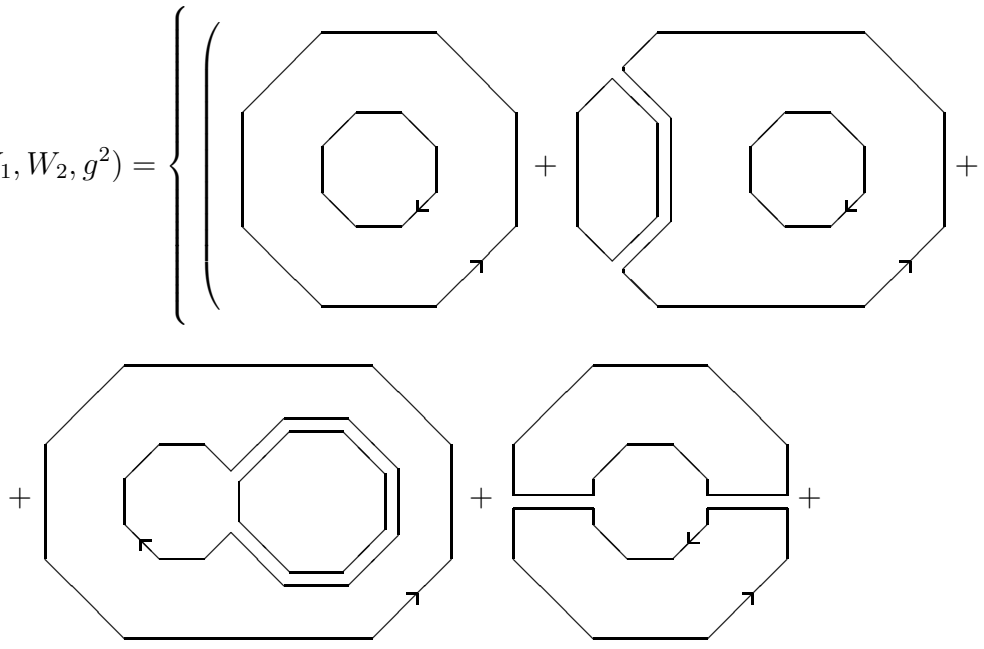
We can now sketch a very rough *direct* proof that the right-hand side of the group-variation equation for each f_0 is indeed equal to the same sum of Feynman diagrams, with the same coefficients, as the left-hand side. Consider any *Feynman* diagram that contributes to $f_0(W_1, \ldots, W_n, g^2)$, and let u be its number of windows. Now the number of ways of colouring each of the u windows of this *Feynman* diagram *independently* in any of M colours is simply M^u. Taking the derivative of this with respect to M at $M = 1$ gives $(uM^{u-1})_{M=1} = u$, which is the coefficient of that Feynman diagram in the left-hand side of the group-variation equation for $f_0(W_1, \ldots, W_n, g^2)$. And, given any colouring of the u windows of this Feynman diagram independently in any of M colours, we erase from that diagram all propagators that have windows of the same colour on each side. The result is, roughly speaking, a valid colouring of one of our right-hand side group-variation equation diagrams, and we immediately see, that out of all the M^u independent colourings of the u windows of that Feynman diagram with M colours, the number of times we get that particular group-variation diagram is precisely given by the chromatic polynomial, $\mathbf{C}(M)$, for that diagram, so that, taking the derivative with respect to M at $M = 1$, the result is precisely $\frac{d}{dM}\mathbf{C}(M)\Big|_{M=1}$, in agreement with the group-variation equations. This argument is of course incomplete since we have totally ignored the kinematic and Lorentz-index structure of the vertices. We also have to consider all the distinct ways that the cubic and quartic vertices arise from the sums over paths given by the group-variation equation gauge-field and

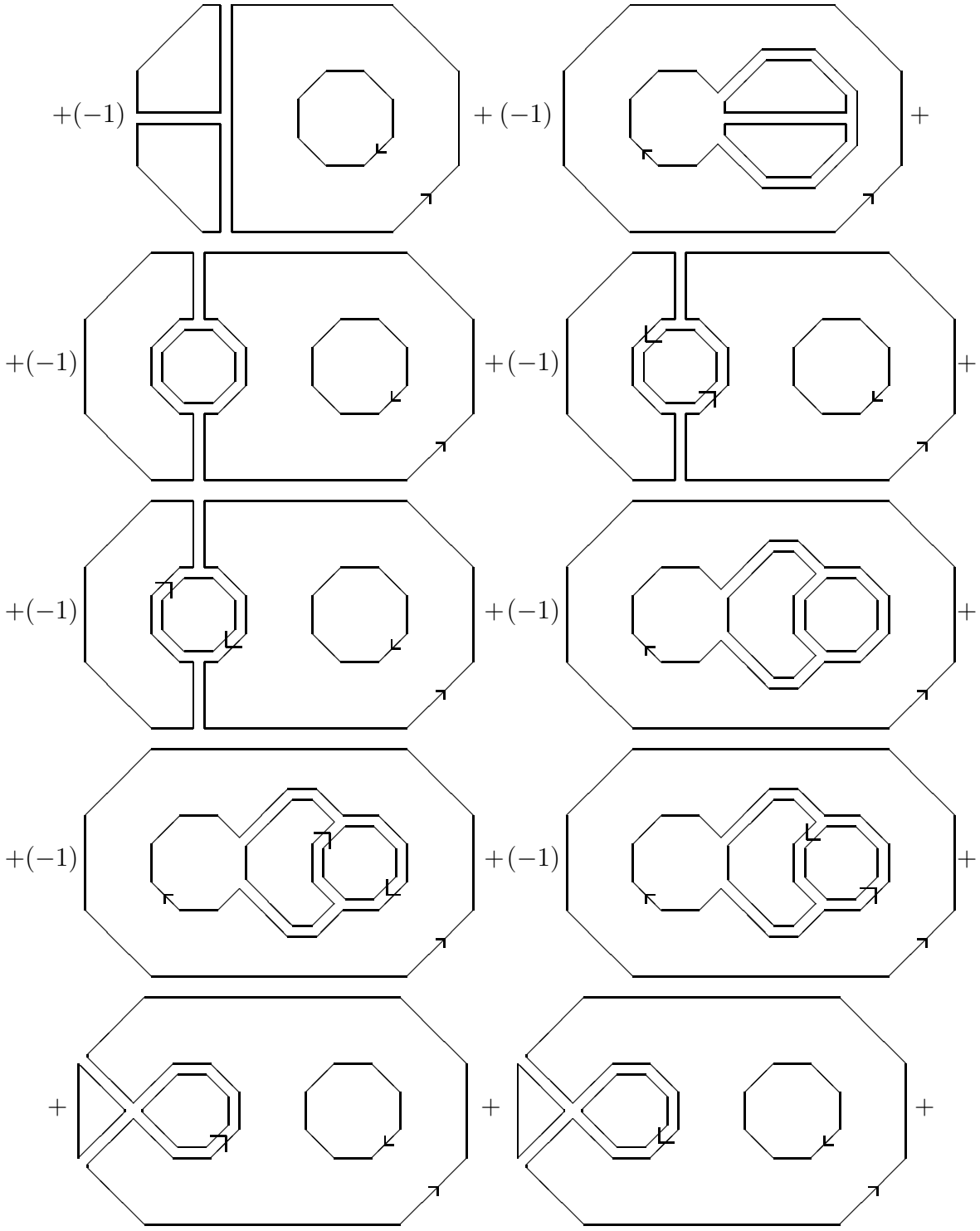
Fadeev-Popov propagators, for example four-gluon vertices arise firstly from $\sigma \to 0$ contributions to (1.27) where one or more of the individual short straight segments have *two* $A_{\mu a}$'s each on them, (which gives the AA terms in (1.28)), secondly from $F_{\mu\nu}$ insertions as given by (1.18), and thirdly from $\frac{1}{E}$ as given by (1.16), (1.20), and (1.30). (The $\sigma \to 0$ limit of (1.27) is of course dominated by contributions where most of the straight segments have *no* $A_{\mu a}$'s on them, some have one $A_{\mu a}$ on them and some have two $A_{\mu a}$'s on them.) Finally we use Mills-type [34] propagator gauge invariance identities ot transform the sums of Feynman diagrams we get in the right-hand sides of the group-variation equations, which involve extra vertices corresponding to our choice (1.6) of gauge-fixing action, and the corresponding Fadeev-Popov action (1.7), to sums of ordinary Feynman diagrams. For full details of all this see the next paper in this series.

4.4 The Group-Variation Equation for the correlation function of two Wilson loops

The explicit form of the group-variation equation for $f_0(W_1, W_2, g^2)$ is:

$$g^2 \frac{d}{dg^2} f_0(W_1, W_2, g^2) =$$

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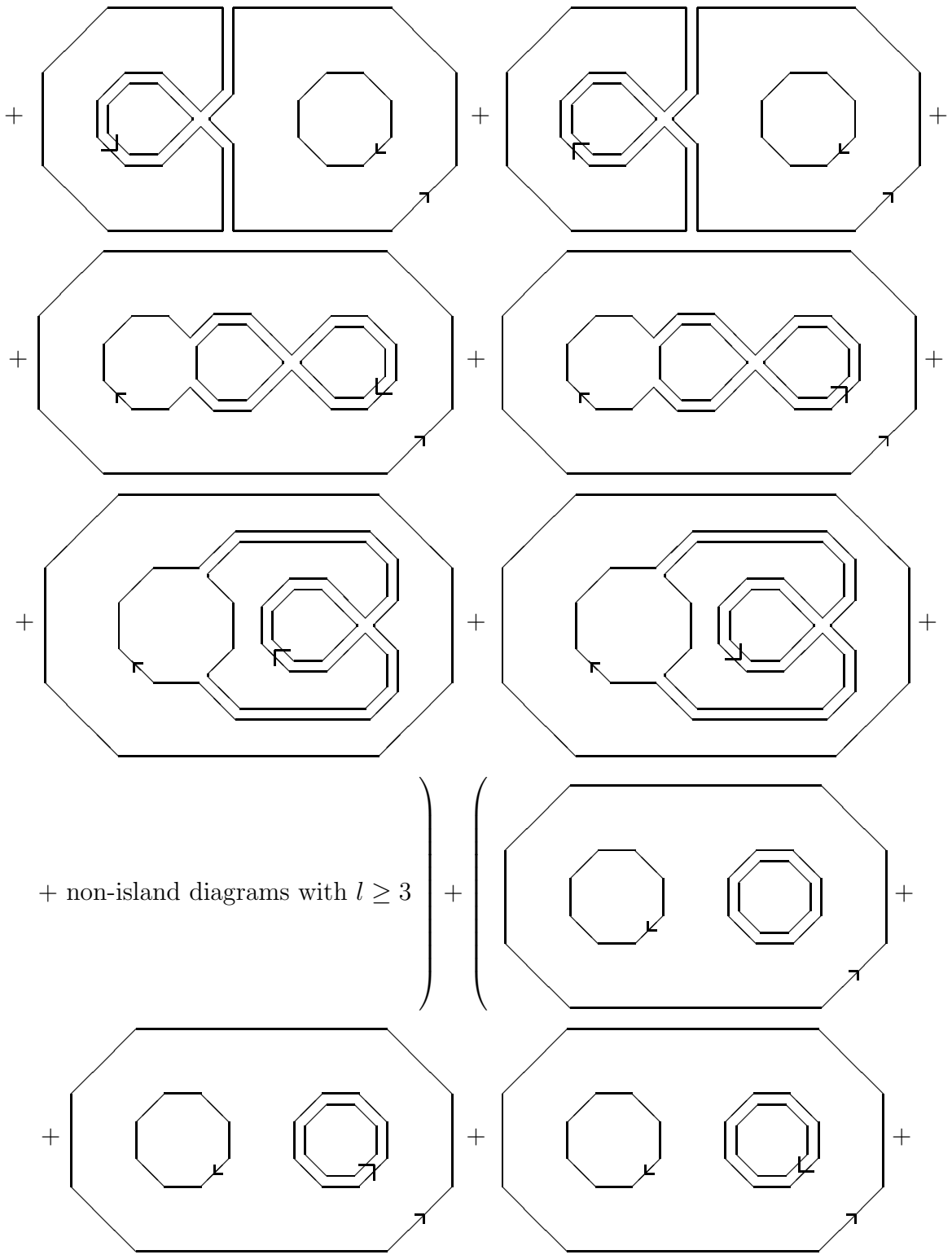
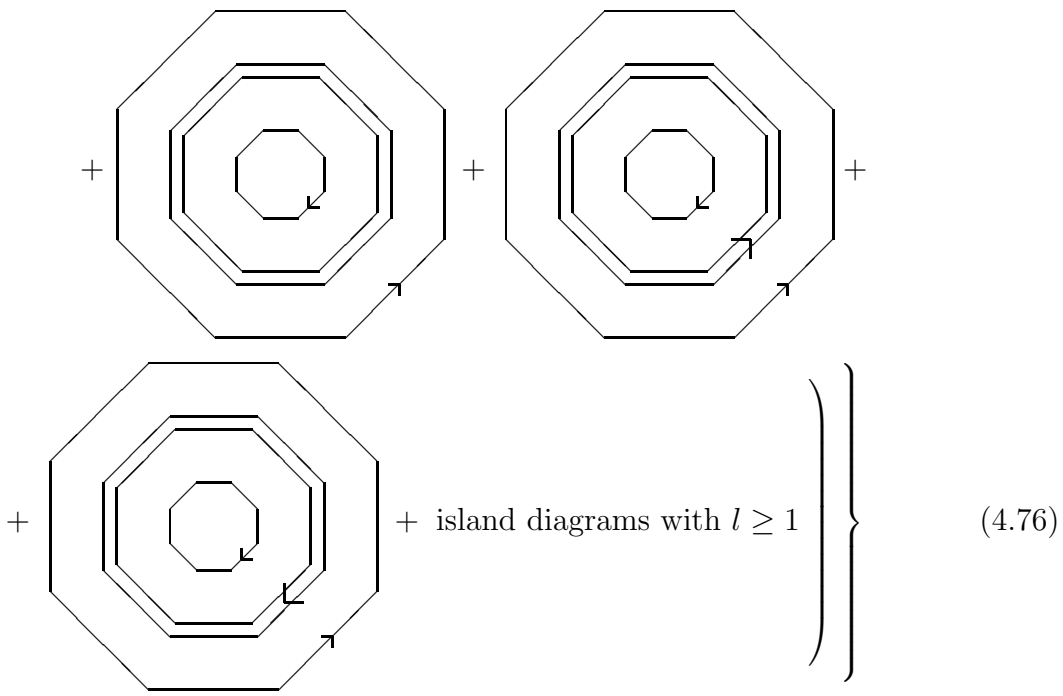

+ non-island diagrams with $l \geq 3$



(4.76)

4.5 Effective mass for the 45-paths, from the window weights in the path integrals

We now outline a demonstration that the solutions of the group-variation equations have the correct qualitative behaviour, namely the Wilson area law [38] for the vacuum expectation value of one Wilson loop, and massive glueball saturation of the correlation functions of two or more Wilson loops.

Let us *suppose* that the vacuum expectation values of large Wilson loops have the qualitative behaviour $e^{-\mu^2 A}$, where μ is a fixed mass and A is the area of the minimal-area orientable spanning surface of the loop [39]. What consequences will this have for the sums over 45-paths bordered by simply-connected windows in the right-hand sides of the group-variation equations? It is clear that the sums over paths will be suppressed in comparison to the "free-path" case (i.e. with $\frac{-1}{\partial^2}$ given by the $\sigma \to 0$ limit of (1.27) with $(W_{xz_1z_2\ldots z_ny})_{AB}$ removed), since the "free-path" weight-factor of every path is now multiplied by an additional factor whose magnitude is ≤ 1, and whose magnitude gets smaller and smaller, as the area of the minimal-area spanning surface of the paths around the edge of any simply-connected window, gets larger and larger.

4.5.1 Quantitative estimate of the effective mass

To get a first quantitative estimate of this effect we consider a 45-path whose ends are well separated, and which has on each side, a simply-connected window, such that all the vertices of those two windows are well separated from one another, and all lie roughly in a single two-dimensional plane. Let the ends of the 45-path of interest be at x and y, and let B denote the 2-plane in which our two windows roughly lie. We consider paths from x to y whose projections into the 2-plane B follow the straight line from x to y. We also require that in each of the $4 - 2 = 2$ dimensions perpendicular to the 2-plane B, the components of our paths in these two "transverse" dimensions, considered as functions of distance along the straight line from x to y, remain small near x and y, but may become larger, but not too large, between x and y, but well away from both. We choose a coordinate system that has its first axis along the straight line from x to y, and its second axis perpendicular to this line in the 2-plane B, so that the third and fourth axes are perpendicular to the 2-plane B. Then under the conditions stated, and in the approximation of retaining only zeroth, first, and second order terms in the transverse components of our paths, the area of the minimal-area spanning surface of either of our two windows, is equal to the area of the projection into the 2-plane B, (which, under our assumptions, is fixed), plus, for each transversed dimension z, a contribution:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}s\mathrm{d}t \frac{(z(s) - z(t))^2}{4\pi(s-t)^2} \tag{4.77}$$

Here s and t represent distance along the straight line from x to y, and we are able to extend the limits to $\pm\infty$ due to our requirement that z be negligibly small except in the region between x and y and well away from both of them. ((4.77) is given by the solution of Laplace's equation in a half-plane with the boundary function z.)

Our first estimate of the window-weighted path integrals is then based on the following approximations:

(i) We neglect all effects on the window areas of deviations from straightness of the projection of our path into the 2-plane B. This will have *no effect* on the sum of the "planar components" of the two areas, provided the projection of our path has no self-intersections, since whatever is lost by one window is gained by the other. However we may assume that we will *underestimate* the suppression of paths whose projections into the 2-plane B have self-intersections.

(ii) We use the formula (4.77) for the transverse components of the window areas, with the further approximation that the parameters s and t are simply proportional to the number of segments along the path. Thus in term n in (1.27) we remove the factor $(W_{xz_1z_2\ldots z_ny})_{AB}$ and include instead, for each of our two windows and each of the two dimensions perpendicular to the 2-plane B, a factor given by the exponential of:

$$-\frac{\mu^2}{4\pi} \sum_{\substack{1 \le s \le n \\ 1 \le t \le n \\ s \ne t}} \frac{(z_s - z_t)^2}{(s-t)^2} \tag{4.78}$$

This is actually not as bad as it may seem, since we know from Douglas's solution of Plateau's problem [39] that the area of the minimal-area orientable spanning surface of *any* simple closed path may be expressed in the quadratic form (*E187*), summed over *all* d dimensions, (in our case, 4), in which the path exists, provided that the *parametrization* of the path is the special parametrization that satisfies Douglas's variational condition [39]. This implies, in particular, that for any *planar* simple closed path without self-intersections, there is a parametrization such that the area enclosed by the path is given by (4.77), summed now over the two dimensions *in* the plane of the path. This parametrization must be the *same* as that for which (4.77), summed over the $4 - 2 = 2$ dimensions *perpendicular* to the base-plane of the path, correctly gives the transverse contributions to the area of the minimal-area spanning surface, for small perturbations of the path perpendicular to its base plane. (Thus for a simple planar closed path, the correct parametrization is given by the conformal transformation that maps the real axis onto the path, for example the Schwarz-Christoffel transformation fi the path cnsists of a finite number of straight segments.) It follows that (4.78) is only wrong to the extent that the parametrization given by simply counting the number of straight segments along the path from x, is not the solution of Douglas's variational condition. And in the central region of the itegration domain of the two components of the z_i's in the 2-plane B, where these two components of the z_i's are distributed uniformly and in order along the straight line from x to y, our parametrization does coincide roughly with the solution of Douglas's variational condition. (Douglas's variational condition is simply the requirement that the parametrization of the path be such that the sum of (4.77) over *all* d dimensions, (i.e. 4 dimensions in our case), in which the path exists, has the minimum possible value among all parametrizations of

the path. The real axis may of course be conformally mapped to the unit circle if required by $s \to \frac{s-i}{s+i}$.)

(iii) Our third approximation is that we treat the effects of the ends of our path in the simplest possible way.

Now we readily find, by induction on n, that for all $n \geq 0$:

$$\int \mathrm{d}z_1 \ldots \int \mathrm{d}z_n \frac{e^{-\frac{(x-z_1)^2}{4\sigma}}}{\sqrt{4\pi\sigma}} \frac{e^{-\frac{(z_1-z_2)^2}{4\sigma}}}{\sqrt{4\pi\sigma}} \ldots \frac{e^{-\frac{(z_n-y)^2}{4\sigma}}}{\sqrt{4\pi\sigma}} = \frac{e^{-\frac{(x-y)^2}{4\sigma(n+1)}}}{\sqrt{4\pi\sigma(n+1)}} \tag{4.79}$$

Now if we set $x = y = 0$ in this formula, as is appropriate, in our chosen coordinate system, for the two dimensions perpendicular to the 2-plane B, we may obtain the same result by finding, for $n \geq 1$, the eigenvalues of the quadratic form in the z_s's, $1 \leq s \leq n$, that occurs in the exponent in the left-hand side.

Indeed, the normalized eigenstates of this quadratic form are given by:

$$z_s = \sqrt{\frac{2}{(n+1)}} \sin qs \qquad\qquad 1 \leq s \leq n \tag{4.80}$$

where

$$q = \frac{m\pi}{n+1} \qquad\qquad 1 \leq m \leq n \tag{4.81}$$

The corresponding eigenvalues are given by:

$$\frac{(1-\cos q)}{2\sigma} = \frac{1}{\sigma} \sin^2\left(\frac{q}{2}\right) \tag{4.82}$$

We do indeed find that the product, over all the eigenvalues λ, of $\sqrt{\frac{\pi}{\lambda}}$, is equal to $\frac{(4\pi\sigma)^{\frac{n}{2}}}{\sqrt{n+1}}$, obtaining incidentally, for all integers $n \geq 1$, the identities:

$$\prod_{m=1}^{n} \sin\left(\frac{m\pi}{2(n+1)}\right) = 2^{-n}\sqrt{n+1} \tag{4.83}$$

We shall estimate the effect of including in the integrand of (4.79), a factor given by the exponential of (4.78) for the window on each side of our path, (or in other words, a net factor given by the exponential of twice (4.78)), by calculating, in first order perturbation theory, the modifications to the eigenvalues (4.82) produced by including twice (4.78) in the exponent, and we readily find, in the approximation of neglecting the effects of the ends of the path, that (4.82) is modified by the addition of:

$$\frac{2\mu^2}{\pi}\left((1-\cos q) + \frac{1}{4}(1-\cos 2q) + \frac{1}{9}(1-\cos 3q) + \ldots\right) \tag{4.84}$$

(In fact, in this approximation, the eigenstates (4.80), (4.81) are also eigenstates of (4.78).)

Now (4.84) is equal to:

$$\mu^2 \left(|q| - \frac{q^2}{2\pi} \right) \tag{4.85}$$

for all q such that $-2\pi \leq q \leq 2\pi$. However the convergence is very slow for small $|q|$, (and for q near -2π and near 2π), hence, bearing in mind that the relevant values (4.81) of q all lie between 0 and π, we may expect a problem, associated with our neglect of the details at the ends of the path, for $q \to 0$.

Now adding (4.85) to (4.82) is equivalent to multiplying (4.82) by:

$$\left(1 + \frac{\mu^2 \sigma \left(|q| - \frac{q^2}{2\pi} \right)}{\sin^2 \left(\frac{q}{2} \right)} \right) \tag{4.86}$$

Now in the notation of equations (1.25) and (1.26), $(n+1)\sigma = s$, where s is the integration variable in equation (1.24). We are interested in the limit $\sigma \to 0$ with s fixed, so the product of (4.86) over all the values of q, as given by (4.81), exponentiates. Thus multiplication of the integrand of (4.79), which refers to *one* transverse dimension, by the exponential of (4.78), results in multiplying the "free path" factor

$$\left(e^{s\partial^2} \right)_{x,y} = \frac{e^{-\frac{(x-y)^2}{4s}}}{(4\pi s)^2}$$

in the integrand of (1.24) by:

$$\exp \left(-\frac{\mu^2 s}{2\pi} \sum_{m=1}^{n} \left(\frac{\pi}{n+1} \right) \left(\frac{\left(|q| - \frac{q^2}{2\pi} \right)}{\sin^2 \left(\frac{q}{2} \right)} \right) \right) \qquad \to$$

$$\to \qquad \exp \left(-\frac{\mu^2 s}{2\pi} \int_0^\pi \mathrm{d}q \left(\frac{\left(|q| - \frac{q^2}{2\pi} \right)}{\sin^2 \left(\frac{q}{2} \right)} \right) \right) \tag{4.87}$$

Now we have $4-2=2$ transverse dimensions, hence within our approximations the effect of our window weights on our sum over paths is simply to multiply the free path factor in the integrand of equation (1.24) by the square of (4.87). This means that the integrand of the s integral in (1.24) has now become precisely that for a *massive*, *free* scalar particle, with mass squared given by:

$$\frac{\mu^2}{\pi} \int_0^\pi \mathrm{d}q \left(\frac{\left(|q| - \frac{q^2}{2\pi} \right)}{\sin^2 \left(\frac{q}{2} \right)} \right) \tag{4.88}$$

Now as we anticipated, there is a problem for $q \to 0$: the integral diverges logarithmically. This is because the sum in (4.84) should have been cut off after about n terms due to the ends of the path. This does not matter for q not close to 0, since the sum converges well for q not close to 0, (or an integer multiple of 2π), but the sum converges very slowly for q close to 0. (This is a direct consequence of the discontinuity of the derivative of (4.85) at $q = 0$ - the Gibbs phenomenon.) Therefore the integrand in (4.88) should be smoothly cut off as $q \to 0$. Now the integrand in (4.88) is well approximated for all $0 < q < 2\pi$ by:

$$\frac{4}{q} + \frac{4}{2\pi - q} - \left(\frac{8}{\pi} - \frac{\pi}{2}\right) \tag{4.89}$$

If we replace the term $\frac{4}{q}$ by $\frac{4}{\epsilon}$ for $0 \leq q \leq \epsilon$ then the integral in (4.88) becomes:

$$4 + 4\ln\left(\frac{\pi}{\epsilon}\right) + 4\ln 2 - \left(8 - \frac{\pi^2}{2}\right) = 4\ln\left(\frac{\pi}{\epsilon}\right) + 3.7 \tag{4.90}$$

For example, $\epsilon = 1$ gives 8.3, hence an effective mass 1.6μ, while $\epsilon = \frac{1}{4}$ gives 13.8, hence an effective mass 2.1μ.

For q away from 0, the integrand in (4.88) is not sensitive to the effects of the ends of the path, hence since the integrand in (4.88) decreases monotonically from $q = 0$ to $q = \pi$, with the derivative being zero at $q = \pi$, and the value $\frac{\pi}{2}$ of the integrand at $q = \pi$ being the absolute minimum value of the integrand function for all $-2\pi \leq q \leq 2\pi$, a conservative estimate of the effective mass is given by replacing the integrand in (4.88) by its minimum value $\frac{\pi}{2}$ for all $0 \leq q \leq \pi$, which gives the value $\mu\sqrt{\frac{\pi}{w}} = 1.3\mu$.

The important point is that the predominant effect, on the right-hand sides of the group-variation equations, of a qualitative behaviour $e^{-\mu^2 A}$ of the vacuum expectation values of large Wilson loops, where A is the area of the minimal-area orientable spanning surface of the loop, is that for each 45-path such that at least one of the two windows beside it, is simply connected, the sum over paths is approximately the free propagator for a *massive* particle, with mass approximately μ, between those two vertices, *and thus is suppressed exponentially, by a factor* $e^{-\mu r}$, *for* $r \geq \mu$, *where* r *is the distance between the ends of the path.*

4.6 Consequences of the effective mass, for different types of diagrams

Now in the group-variation equation (4.74) for $f_0(W_1, g^2)$, *every* 45-path has at least one simply-connected window beside it, hence every path brings in a factor $e^{-\mu r}$, where r is the distance between the ends of the path. For the one-loop islands we may fix two or three points on the path, which corresponds to representing $e^{s\bar{D}^2}$ in (1.33) as $e^{\frac{s}{3}\bar{D}^2}e^{\frac{s}{3}\bar{D}^2}e^{\frac{s}{3}\bar{D}^2}$ (for three points fixed), and we again obtain, for each pair of consecutive fixed points around the path, an exponential suppression factor $e^{-\mu r}$, where r is the distance between those two fixed points. Thus when the size of the loop W_1 is larger than $\frac{1}{\mu}$, the predominant contributions to the right-hand side of the group-variation equation (4.74) for $f_0(W_1, g^2)$ come from, firstly, non-island diagrams, whose one band is roughly of size $\frac{1}{\mu}$, and thus creeps like a blob along the loop W_1, and secondly, from *island* diagrams, with the size of the island being roughly μ. (Of course, as we have already noted, the first term in the right-hand side, which has no band or island, is simply $f_0(W_1, g^2)$, amd cancels the $f_0(W_1, g^2)$ term in the left-hand side.)

Furthermore, looking at the right-hand side of the group-variation equation (4.76) for $f_0(W_1, W_2, g^2)$, we see that apart from the first term, which is simply $f_0(W_1, W_2, g^2)$ itself, there are, firstly, non-island diagrams with exactly one band, with all the ends of that band being on *one* of the two loops W_1 and W_2. Every 45-path in such a band has at least one simply-connected window beside it, and thus the predominant contributions of such diagrams come from a band of size $\frac{1}{\mu}$, which thus, for two large and well-separated loops, creeps like a blob along the loop to which it is attached. Secondly, there are non-island diagrams with one or more bands, each of which has at least one end on *both* the loops W_1 and W_2. (The simplest such diagram with exactly one band has $l = 3$.) In such diagrams *every* window is simply-connected, hence again the predominant contributions of such diagrams come from bands of size $\frac{1}{\mu}$. Thus if we define r to be the smallest distance between any point on W_1, and any point on W_2, and r is greater than $\frac{1}{\mu}$, then the total contributions of such diagrams are suppressed by an exponential factor $e^{-\mu r}$. Thirdly there are island diagrams where, just as in the corresponding diagrams in the group-variation equation for $f_0(W_1, g^2)$, among all the windows beside paths of the island, exactly *one* of those windows is *not* simply-connected. We call island diagrams of this type *Type-1 island diagrams.* Every 45-path of such an island has at leat one simply-connnected window beside it, hence

the predominant contributions of such island diagrams, when at least one of W_1 and W_2 and r, (the shortest distance between W_1 and W_2), is largee, come from islands of size μ. And finally there are island diagrams of a new sort, the simplest being example (4.55), where among all the windows beside paths of the island, two or more of those windows are *not* simply-connected. We call all island diagrams of this type *Type-2 island diagrams.*

We note that for all $n \geq 1$, it is precisely the Type-1 island diagrams that result in $f_0(W_1, \ldots, W_{n+1}, g^2)$ occurring in the right-hand side of the group-variation equation for $f_0(W_1, \ldots, W_n, g^2)$. Indeed, in a Type-1 island diagram in the right-hand side of the group-variation equation for $f_0(W_1, \ldots, W_n, g^2)$, the one window beside paths of the island that is *not* simply connected, is topologically a sphere with $n+1$ holes, (or a disk with n holes), and thus corresponds to $f_0(W_1 \ldots, W_{n+1}, g^2)$. And in every diagram in the right-hand side of the group-variation equation for $f_0(W_1, \ldots, W_n, g^2)$ that is *not* a Type-1 island diagram, every window is topologically a disk with at most $n-1$ holes, and thus corresponds to $f_0(W_1, \ldots, W_m, g^2)$, where $1 \leq m \leq n$. Thus although the group-variation equations for the f_0's do indeed couple together the various f_0's, the mixing is of a very simple form: in the right-hand side of the group-variation equation for $f_0(W_1, \ldots, W_n, g^2)$, the *only* $f_0(W_1, \ldots, W_m, g^2)$ with $m > n$ that occurs is $f_0(W_1, \ldots, W_{n+1}, g^2)$, and this *only* occurs in the Type-1 island diagrams. Furthermore, the dependence of the right-hand side of the group-variation equation for $f_0(W_1, \ldots, W_n, g^2)$ on $f_0(W_1, \ldots, W_{n+1}, g^2)$, is *linear*: in terms of $f_0(W_1, \ldots, W_{n+1}, g^2)$, the right-hand side of the group-variation equation for $f_0(W_1, \ldots, W_n, g^2)$, is equal to a term independent of $f_0(W_1, \ldots, W_{n+1}, g^2)$, plus a term linear in $f_0(W_1, \ldots, W_{n+1}, g^2)$.

4.7 The u and d quark static masses

We note in passing that the quantitative value of the effective mass produced by the window weights with just *one* window beside the path is in fact of great interest, since it determines the effective quark mass to be used in static quark calculations, (e.g. Ref [40]). The effective mass squared for $4-2=2$ transverse dimensions and *one* window beside the path is half (4.88), hence if, as suggested, we replace the integrand of (4.88) by its minimum value of $\frac{\pi}{2}$ for all $0 \leq q \leq \pi$, we obtain $\mu\sqrt{\frac{\pi}{4}} = 0.89\mu$. Experimentally [41], μ is equal to $\frac{1}{\sqrt{2\pi\alpha'}} = 0.41$ GeV, where $\alpha' = 0.93$ GeV^{-2} is the universal Regge slope, from which we obtain $\mu\sqrt{\frac{\pi}{4}} = 0.37$ GeV, in agreement with the observed u and

d static quark masses [40].

4.8 Qualitative behaviour of the correlation functions

We now develop a hypothesis for the qualitative behaviour of $f_0(W_1, \ldots, W_n, g^2)$ for $n \geq 2$, in preparation for stating, in the next chapter, our ansatz for the correlation functions, which we will show, in Chapter 7, is consistent with the group-variation equations, once it has been modified, by the inclusion of appropriate pre-exponential factors, for the terms in the correlation functions.

4.8.1 Minimal-area spanning surfaces of higher topology

Let the sizes and separations of $W_1, \ldots, W_n$ all be $\geq \frac{1}{\mu}$.

Let S be the minimal-area orientable spanning surface of $W_1, \ldots, W_n$. Formally, S is defined to be the 2-dimensional manifold which, among all the measurable orientable 2-dimensional manifolds with boundary $W_1, \ldots, W_n$, is the one with the smallest possible area. (The orientations of the manifolds considered are required to agree with the given orientations of $W_1, \ldots, W_n$.) S is *not* required to be connected. Courant [39] has shown that an S realizing the minimum possible area always exists. Usually S is unique, but in certain cases there may be two or more different surfaces S which have the minimum possible area. Our loops $W_1, \ldots, W_n$ are always defined by a finite number of parameters, (in practice they are always formed from a finite number of straight segments, but we may consider circles in examples). The degenerate cases occur when $W_1, \ldots, W_n$ has two or more different *locally* minimal spanning surfaces, in the sense of having everywhere vanishing mean curvature, and as the parameters defining $W_1, \ldots, W_n$ are varied continuously, a transition takes place between which of the different *locally* minimal spanning surfaces, is the one which gives the absolute minimum value of the area. Thus the degenerate cases occur at a set of points of measure zero in the finite-dimensional space of the parameters defining $W_1, \ldots, W_n$, and we may for practical purposes assume that S is unique.

For example, suppose that $n = 2$ and that we have two circles of equal radius r, lying in a common 3-dimensional subspace of our 4-dimensional Euclidean space, such that both circles are perpendicular to the straight line between their centres. Let the

two circles have *opposite* orientations, (so that, if view as wheels, they would rotate in opposite directions about their common axis), and let d be the distance between their centres. Then one locally minimal oriented spanning surface of W_1 and W_2, which exists for all $d > 0$, is simply the union of the flat disks which fill the two circles. It has two connected components, and its area is $2\pi r^2$. And for $d \leq 1.3255r$ there is also a *locally* minimal oriented spanning surface which has the topology of a cylinder, (i.e. a sphere with two holes). This surface is a surface of revolution about the common axis of the two "wheels". It is described by:

$$s = \frac{1}{\alpha}\cosh \alpha t \tag{4.91}$$

where t denotes the distance along the common axis of the two "wheels" from the point midway between them, s denotes the radial distance perpendicular to this axis, and α is determined by:

$$\alpha r = \cosh\left(\alpha \frac{d}{2}\right) \tag{4.92}$$

The solution of (4.92) may be found graphically by superimposing, on the graph of the curve $y = \cosh x$, the straight line through the origin with slope $\frac{2r}{d}$. If the straight line cuts the curve at (x, y), $x > 0$, then a solution of (4.92) is given by $\alpha = \frac{2x}{d}$. We see that if $\frac{2r}{d} \geq 1.5089$, (where $1.5089 = \frac{\cosh 1.1997}{1.1997}$, and 1.1997 is the positive solution of $x \sinh x = \cosh x$), then (4.92) in fact has *two* solutions, each of which gives a *locally* minimal orientable spanning surface of W_1 and W_2. However for $\frac{2r}{d} > 1.5089$ the solution with the larger value of x, and hence the larger value of α, always has larger area than the other one, and hence may be discarded. For $\frac{2r}{d} = 1.5089$ the two solutions coincide, while for $\frac{2r}{d} < 1.5089$ there are *no* locally minimal orientable spanning surfaces with cylinder topology: the straight line misses the curve $y = \cosh x$ altogether.

Furthermore, even when the locally minimal surface of cylinder topology does exist, its area is not necessarily smaller than the area of the two disks. Its area is:

$$\int_{-\frac{d}{2}}^{\frac{d}{2}} 2\pi s \sqrt{1 + \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \frac{\pi}{\alpha}\left(d + \frac{\sinh \alpha d}{\alpha}\right) \tag{4.93}$$

where α is determined, for $\frac{2r}{d} \geq 1.5089$, or in other words, for $d \leq 1.3255r$, as the smaller of the two positive solutions of (4.92). For small $\frac{d}{r}$, the area is approximately equal to $2\pi r d$, and for fixed r, it is a monotonically increasing function of d. It becomes

equal to $2\pi r^2$ for $d = 1.0554r$, (where $1.0554 = \frac{2 \times 0.6392}{\cosh 0.6392}$, and 0.6392 is the solution of $\cosh^2 x = x + \frac{1}{2} \sinh 2x$).

Thus in the present example, our surface S, which by definition is the *absolute* minimal-area orientable spanning surface of the two circles, is equal to the cylinder-topology locally minimal spanning surface, as just constructed, for $d < 1.0554r$, and equal to the union of the flat disks which fill the two circles, for $d > 1.0554r$. The degenerate case, where S could be either the cylinder-topology surface or the union of the two flat disks, occurs only for $d = 1.0554r$ which defines a measure-zero subspace of the two-dimensional space of the parameters d and r.

We note that if we had chosen the two circles to have the *same* orientation, (so that, viewed as wheels, they would rotate in the *same* direction about their common axis), then S, the absolute minimal area orientable spanning surface of the two circles, would have been the union of the two flat disks for *all* $d > 0$.

4.8.2 Minimal-length spanning trees

Returning now to the general case of $W_1, \ldots, W_n$, let $S_1, \ldots, S_m$, where $1 \leq m \leq n$, denote the separate connected components of our absolute minimal-area orientable spanning surface S. We now require to find the "tree" of straight line segments, of the smallest possible total length, such that the ends of the straight line segments are either at junctions of the tree or on connected components of S, and such that the union of S and all the straight line segments is *connected*.

For example, if $m = 1$, so that S is itself connected, we do not need any straight line segments at all. If $m = 2$, we have exactly one straight line, which has one end on S_1 and one end on S_2. It is the shortest possible straight line segment that has one end on S_1 and one end on S_2. (There may, once again, in some cases be a degeneracy, even a continuous degeneracy, among different possible such shortest line segments. This occurs, for example, in the above example of two circles in any axisymmetric configuration within a three-dimensional subspace of our four-dimensional space, when the absolute minimal-area orientable spanning surface of the two circles is the union of the two flat disks. Such cases, once again, occur only in domains of measure zero in the finite-dimensional spaces of the parameters of our loops. Furthermore, our ansatz will depend, firstly, on the total length of all the straight line segments, and secondly, on the lengths of the individual straight line segments, but not on the precise positions of

the individual line segments.) If $m = 3$, we may either have two straight line segments, each of which has its ends on two different members of $\{S_1, S_2, S_3\}$, (in which case one of S_1, S_2, and S_3 has two straight lines ending on it, and the other two each have one straight line ending on them), or else we may have three straight lines, each of which has one end on one of S_1, S_2, and S_3, and its other end at a junction where all three straight lines meet.

Such "minimal-length spanning trees" have, as is well known, the following properties:

(i) At most three straight lines meet at any junction of straight lines.

(ii) At any junction where three straight lines meet, all three straight lines lie within a single 2-plane, and the angle between each pair of lines is $\frac{2\pi}{3}$.

Indeed, let us consider any junction where exactly three straight lines meet. In general, the position of the junction, plus the positions of the other ends of each of the three straight lines, defines a three-dimensional space. Within this three-dimensional space, we project the junction perpendicularly onto the two-dimensional plane defined by the outer ends of the three straight lines, and if the junction was not in this two-dimensional plane to start with, we thereby obtain a new position of the junction giving a strictly smaller total length of the three straight lines. Thus we may assume that the junction lies within the two-dimensional plane defined by the outer ends of the three straight lines. Let the angles made at the junction by the second and third straight lines, measured in the same sense from the first straight line, be ϕ and ψ respectively. Let us now consider an alternative position of the junction, obtained by moving the old junction a distance ϵ along the straight line in this two-plane starting at the old junction, and makin an angle θ with the first straight line, measured in the same sense as before. Then if the lengths of the original straight lines were p, q, and r, respectively, the lengths of the new straight lines are $\sqrt{p^2 + \epsilon^2 - 2p\epsilon\cos\theta}$, $\sqrt{q^2 + \epsilon^2 - 2q\epsilon\cos(\theta - \phi)}$, and $\sqrt{r^2 + \epsilon^2 - 2r\epsilon\cos(\theta - \psi)}$, respectively. Thus by expanding in powers of ϵ, and dropping terms of second and higher order in ϵ, we find that the increase in length, for small ϵ, is:

$$-\epsilon(\cos\theta + \cos(\theta - \phi) + \cos(\theta - \psi)) =$$

$$= -\epsilon((1 + \cos\phi + \cos\psi)\cos\theta + (\sin\phi + \sin\psi)\sin\theta) \tag{4.94}$$

This has the form:

$$-\epsilon\xi\cos(\theta - \zeta) \tag{4.95}$$

where

$$\xi^2 = (1 + \cos\phi + \cos\psi)^2 + (\sin\phi + \sin\psi)^2 \tag{4.96}$$

and

$$\tan\zeta = \frac{\sin\phi + \sin\psi}{1 + \cos\phi + \cos\psi} \tag{4.97}$$

Thus we can always find a θ which results in a reduction in the total length, unless $\xi = 0$. Now $\xi = 0$ implies:

$$1 + \cos\phi + \cos\psi = \sin\phi + \sin\psi = 0 \tag{4.98}$$

And $\sin\phi + \sin\psi = 0$ implies either $\psi = -\phi$ or $\psi = \pi + \phi$, and $\psi = \pi + \phi$ implies $\cos\psi = -\cos\phi$, hence cannot be the solution. Hence $\psi = -\phi$, hence $\cos\psi = \cos\phi = -\frac{1}{2}$, hence $\phi = \pm\frac{2\pi}{3}$.

We now consider three points a, b, and c, and identify their minimal-length spanning tree. We define:

$$\alpha \equiv (b-c)^2,\ \beta \equiv (c-a)^2,\ \gamma \equiv (a-b)^2,\ \eta \equiv \frac{1}{2}(\alpha+\beta+\gamma),$$

$$A \equiv \frac{1}{4}\sqrt{2(\beta\gamma+\gamma\alpha+\alpha\beta) - (\alpha^2+\beta^2+\gamma^2)}$$

$$= \text{area of triangle abc},$$

$$\mathcal{X} \equiv \eta + 2\sqrt{3}A,$$

$$\mathcal{A} \equiv \sqrt{3}(\eta-\alpha) + 2A = \sqrt{3}(\mathcal{X}-\alpha) - 4A,$$

$$\mathcal{B} \equiv \sqrt{3}(\eta-\beta) + 2A = \sqrt{3}(\mathcal{X}-\beta) - 4A,$$

$$\mathcal{C} \equiv \sqrt{3}(\eta-\gamma) + 2A = \sqrt{3}(\mathcal{X}-\gamma) - 4A,$$

$$\mathcal{Z} \equiv \frac{\mathcal{A}\mathcal{B}\mathcal{C}}{4A\sqrt{3}\mathcal{X}},\ s \equiv \frac{\mathcal{Z}}{\mathcal{A}},\ t \equiv \frac{\mathcal{Z}}{\mathcal{B}},\ u \equiv \frac{\mathcal{Z}}{\mathcal{C}} \tag{4.99}$$

and note the identities:

$$(\eta-\beta) + (\eta-\gamma) = \alpha,\ \text{etc.},$$

$$A = \frac{1}{4}\sqrt{2(\eta-\alpha)\alpha + 2(\eta-\beta)\beta + 2(\eta-\gamma)\gamma}$$

$$= \frac{1}{2}\sqrt{(\eta-\beta)(\eta-\gamma) + (\eta-\gamma)(\eta-\alpha) + (\eta-\alpha)(\eta-\beta)}$$

$$= \frac{1}{2}\sqrt{\beta\gamma - (\eta-\alpha)^2},\ \text{etc.},$$

$$\mathcal{A} + \mathcal{B} + \mathcal{C} = \sqrt{3}\mathcal{X},$$

$$\mathcal{BC} + \mathcal{CA} + \mathcal{AB} = 4A\sqrt{3}\mathcal{X}, \qquad s + t + u = 1,$$
$$\mathcal{A}^2 + \mathcal{B}^2 + \mathcal{C}^2 = 3\mathcal{X}^2 - 8A\sqrt{3}\mathcal{X} \tag{4.100}$$

We note that it immediately follows from $(\eta - \beta) + (\eta - \gamma) = \alpha$, etc., that the sum of any two of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, is ≥ 0, hence that at most one of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, can be < 0. Furthermore, if $\widehat{cab}$ denotes the angle between the lines $\overline{ab}$ and $\overline{ac}$, then $\cos\widehat{cab}$ is greater than or less than $-\frac{1}{2}$, according as $(\beta + \gamma - \alpha)$ is greater than or less than $-\sqrt{\beta\gamma}$,hence according as $2(\alpha - \eta)$ is less than or greater than $\sqrt{\beta\gamma}$, hence according as $4(\alpha - \eta)^2$ is less than or greater than $\beta\gamma$, hence according as $2A$ is greater than or less than $\sqrt{3}(\alpha - \eta)$, hence according as $\mathcal{A}$ is greater than or less than 0. Hence $\mathcal{A}$ is greater than or less than 0 according as $\widehat{cab}$ is less than or greater than $\frac{2\pi}{3}$. And similarly, $\mathcal{B}$ is greater than or less than 0 according as $\widehat{abc}$ is less than or greater than $\frac{2\pi}{3}$, and $\mathcal{C}$ is greater than or less than 0 according as $\widehat{bca}$ is less than or greater than $\frac{2\pi}{3}$. (of course, at most one of the three angles of the triangle abc can be greater than $\frac{2\pi}{3}$.) We assume first that the angles $\widehat{cab}$, $\widehat{abc}$, and $\widehat{bca}$ are all greater than zero, so that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are all greater than zero. Then the position x of the junction is given by:

$$x = sa + tb + uc \tag{4.101}$$

where s, t, and u are as defined in (4.99). The identity $s + t + u = 1$ in (4.100) means that s, t, and u are the barycentric coordinates of x with reference to a, b, and c, and the assumption that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are all greater than zero implies that s, t, and u are all greater than zero, hence that the junction lies withing the triangle abc. And using (4.99) and (4.100), we readily calculate that:

$$(x - a)^2 = \frac{\mathcal{A}^2}{3\mathcal{X}}, \qquad (x - b)^2 = \frac{\mathcal{B}^2}{3\mathcal{X}}, \qquad (x - c)^2 = \frac{\mathcal{C}^2}{3\mathcal{X}} \tag{4.102}$$

Hence, bearing in mind the assumption that $\mathcal{B}$ and $\mathcal{C}$ are both greater than zero, we find that:

$$\cos\widehat{bxc} = \frac{(x - b)^2 + (x - c)^2 - (b - c)^2}{2\,|x - b|\,|x - c|} = \frac{\mathcal{B}^2 + \mathcal{C}^2 - 3\mathcal{X}\alpha}{2\mathcal{BC}} \tag{4.103}$$

which is readily confirmed, using (4.99) and (4.100), to be equal to $-\frac{1}{2}$. And similarly, we find $\cos\widehat{cxa} = \cos\widehat{axb} = -\frac{1}{2}$. We may also check readily that:

$$\frac{\partial}{\partial x_\mu}(|x - a| + |x - b| + |x - c|) = \frac{(x - a)_\mu}{|x - a|} + \frac{(x - b)_\mu}{|x - b|} + \frac{(x - c)_\mu}{|x - c|} \tag{4.104}$$

does indeed vanish when x takes the value (4.101). We also find from (4.100), that when x takes the value (4.101), the total length of the three straight line segments is given by:

$$|x-a| + |x-b| + |x-c| = \sqrt{\mathcal{X}} \tag{4.105}$$

We may confirm that whenever all three angles $\widehat{cab}$, $\widehat{abc}$, and $\widehat{bca}$ are *strictly* less than $\frac{2\pi}{3}$, $\sqrt{\mathcal{X}}$ is *strictly* less than the sum of the lengths of any two edges of the triangle abc, as follows. To show that $\sqrt{\mathcal{X}}$ is strictly less than $\sqrt{\beta}+\sqrt{\gamma}$, or in other words, that $\mathcal{X}$ is strictly less than $\beta+\gamma+2\sqrt{\beta\gamma}$, we note that the assumption that $\widehat{cab}$ is strictly less than $\frac{2\pi}{3}$ impliess that $\sqrt{\beta\gamma}$ is strictly greater than $\alpha-\beta-\gamma = 2(\alpha-\eta)$. Hence

$$0 < (\sqrt{\beta\gamma} + 2(\eta-\alpha))^2 \tag{4.106}$$

hence

$$3(\beta\gamma - (\eta-\alpha)^2) < (2\sqrt{\beta\gamma} + (\eta-\alpha))^2 \tag{4.107}$$

hence

$$2\sqrt{3}A < \left|2\sqrt{\beta\gamma} + (\eta-\alpha)\right| \tag{4.108}$$

Now by assumption $\frac{1}{2}\sqrt{\beta\gamma} + (\eta-\alpha)$ is greater than zero, hence $2\sqrt{\beta\gamma} + (\eta-\alpha)$ is certainly greater than zero, hence (4.108) implies:

$$2\sqrt{3}A < 2\sqrt{\beta\gamma} + (\eta-\alpha) \tag{4.109}$$

or in other words, by (4.99):

$$\mathcal{X} < \beta+\gamma+2\sqrt{\beta\gamma} \tag{4.110}$$

Thus whenever all three angles $\widehat{cab}$, $\widehat{abc}$, and $\widehat{bca}$ are all strictly less than $\frac{2\pi}{3}$, the minimal-length spanning tree of a, b, and c has a junction strictly in the interior of the triangle abc.

Now if $\widehat{cab}$ is equal to $\frac{2\pi}{3}$, $\mathcal{A}$ is equal to 0, which immediately imples that $s=1$, $t=u=0$, hence that the "junction" x is at a. Thus we see immediately that if any of the angles of the triangle abc is equal to $\frac{2\pi}{3}$, then the minimal-length spanning tree of a, b, and c is equal to the two edges of the triangle abc that meet at the vertex where the angle is $\frac{2\pi}{3}$.

And finally, if any of the angles of the triangle abc is strictly *greater* than $\frac{2\pi}{3}$, then there is *no* point x in the two-plane defined by a, b, and c, such that the angles between the three lines $\overline{xa}$, $\overline{xb}$, and $\overline{xc}$, are all equal to $\frac{2\pi}{3}$. For suppose $\theta \equiv \overline{cab}$ is strictly greater

than $\frac{2\pi}{3}$. We choose a coordinate system in the two-plane defined by a, b, and c, such that $a = (d, e)$, $b = (0, 0,$ and $c = (f, 0)$. Then $\alpha = f^2$, $\beta = (d-f)^2 + e^2$, and $\gamma = d^2 + e^2$. We consider, within this fixed two-plane, with b and c fixed, the locus of all possible positions of a such that θ has the given value. It is determined by:

$$\alpha = \beta + \gamma - 2\sqrt{\beta\gamma}\cos\theta \tag{4.111}$$

which implies:

$$4\beta\gamma\cos^2\theta = (\beta + \gamma - \alpha)^2 \tag{4.112}$$

or in other words:

$$(e^2 + d(d-f))^2\sin^2\theta - e^2f^2\cos^2\theta = 0 \tag{4.113}$$

or in other words:

$$((e^2 + d(d-f))\sin\theta - ef\cos\theta)((e^2 + d(d-f))\sin\theta + ef\cos\theta) = 0 \tag{4.114}$$

which means that the possible positions of $a = (d, e)$ lie on two circles, each of radius $\frac{f}{2\sin\theta}$, one centred at $\frac{f}{2}(1, \cot\theta)$, and the other centred at $\frac{f}{2}(1, -\cot\theta)$. Now we lost the sign of $\cos\theta$ in going from (4.111) to (4.112), and we readily verify that, since $\theta > \frac{\pi}{2}$, the correct arcs of these two circles are the two *shorter* arcs between $b = (0, 0,$ and $c = (f, 0)$. (These are the two arcs which lie inside the compond figure formed by the two intersecting circles.) Thus the possible positions of $a = (d, e)$ lie on these two arcs, which form a convex shape, roughly like a lemon, with endpoints at $b = (0, 0)$ and $c = (f, 0)$. We assume β and γ are both strictly greater than zero, hence the endpoints are excluded. The maximum possible value of e is $\frac{f(1+\cos\theta)}{2\sin\theta} = \frac{f\sin\theta}{2(1-\cos\theta)}$, which for $\frac{\pi}{2} < \theta < \pi$ is a strictly decreasing function of θ. And in exactly the same way we find that the locus of the points x in this two-plane such that $\widehat{bxc} = \frac{2\pi}{3}$, consists of two circular arcs which are obtained from those which give the possible positions of a, by replacing θ by $\frac{2\pi}{3}$. These two arcs form the boundary of a convex lemon-shaped domain which contains all possible positions of a strictly in its interior, (since by assuption a is not equal to either endpoint). It immediately follows from this that there is no point x on these two arcs such that all three of the angles, at x, between the lines $\overline{xa}$, $\overline{xb}$, and $\overline{xc}$, are equal to $\frac{2\pi}{3}$, for when x is not equal to either endpoint, all three of these line segments point into the convex domain bounded by the two arcs, hence the angle between one pair of them is greater than π, while if x is equal to one of the endpoints, then the angle between the line $\overline{xa}$ and the straight line from x to the other endpoint, is

less than $\pi - \theta$, hence less than $\frac{\pi}{3}$. Hence, as stated, if any of the angles of the triangle abc is strictly greater than $\frac{2\pi}{3}$, then there is no point x in the two-plane defined by a, b, and c, such that the angles between the three lines $\overline{xa}$, $\overline{xb}$, and $\overline{xc}$, are all equal to $\frac{2\pi}{3}$. Hence, in this case, the minimal-length spanning tree of a, b, and c cannot have a junction where three lines meet, hence it must consist of two of the edges of the triange abc, and the two shortest edges are the two that meet at the vertex where the angle is greater than $\frac{2\pi}{3}$.

We can now show that at most three straight line segments can meet at any junction in a minimal-length spanning tree. For suppose a minimal-length spanning tree has a junction at which four or more straight line segments meet. Then there must be at least one pair of segments meeting at the junction such that the angle between them at the junction is strictly less than $\frac{2\pi}{3}$. For suppose the angle between every pair of segments at the junction is greater than or equal to $\frac{2\pi}{3}$. We choose a coordinate system with the origin at the junction, and for each i, $1 \leq i \leq n$, where n is the number of line segments meeting at the junction, we define x_i to be the point at unit distance out from the junction along the i$^{\text{th}}$ line segment, (extended if necessary). Then by assumption:

$$x_1^2 = x_2^2 = \ldots = x_n^2 = 1 \tag{4.115}$$

and, for all $1 \leq i < j \leq n$:

$$x_i.x_j \leq -\frac{1}{2} \tag{4.116}$$

Hence

$$(x_1 + x_2 + \ldots + x_n)^2 = n + 2 \sum_{1 \leq i < j \leq n} x_i.x_j \leq \frac{n(3-n)}{2} \tag{4.117}$$

which is impossible for $n > 3$. Hence, as stated, there must be at least one pair of line segments meeting at the junction such that the angle between them at the junction is strictly less than $\frac{2\pi}{3}$. Given such a pair of segments, we form an isosceles triangle, with apex at the junction, by going out from the junction an equal distance, strictly greater than zero, along each of them, without going past the outer end of either segment. The odd angle of this isosceles triangle is strictly smaller than $\frac{2\pi}{3}$ by assumption, hence all its angles are strictly less than $\frac{2\pi}{3}$. Hence, by the foregoing, the minimal-length spanning tree of the vertices of this triangle has a junction strictly inside this triangle, and the total length of the three straight line segments that form this spanning tree, is strictly less than the sum of the lengths of any two edges of this triangle. Hence by replacing the two edges of this triangle that meet at the original junction, by the

minimal-length spanning tree of this triangle, we obtain a spanning tree of strictly smaller total line length than the given spanning tree, and the number of line segments meeting at the given junction has been reduced by one.

Chapter 5

Ansatz For The Vacuum Expectation Values And Correlation Functions, And The Island Diagram Mechanism

5.1 Ansatz for the vacuum expectation values and correlation functions

We can now state our ansatz for the behaviour of $f_0(W_1, \ldots, W_n, g^2)$ for $n \geq 2$, when the sizes and separations of $W_1, \ldots, W_n$ are all $\geq \frac{1}{\mu}$.

5.1.1 The massive scalar propagator

We first note that for $|x-y|$ large compared to $\frac{1}{m}$, the free massive scalar propagator:

$$\int_0^\infty \mathrm{d}s \frac{e^{-\frac{(x-y)^2}{4s}}}{(4\pi s)^2} e^{-m^2 s} \tag{5.1}$$

in our four-dimensional Euclidean space, is found, by a standard steepest descents approximation to the s integral about the peak of the exponential factor at $s = \frac{|x-y|}{2m}$, to approach the asymptotic form:

$$\sqrt{\frac{m}{32\pi^3 |x-y|^3}} e^{-m|x-y|} \tag{5.2}$$

As a check on this, we note that (5.2) immediately gives the Yukawa potential:

$$\frac{e^{-mr}}{4\pi r} \tag{5.3}$$

for the static interaction between two heavy particles, due to exchange of scalar particles of mass m. Indeed, we consider two parallel straight lines separated by a distance r. We put x at a fixed position on one of the two lines, and integrate y along the other line, expanding $|x-y|$ in the exponent as $\sqrt{r^2+z^2} = r + \frac{z^2}{2r}$ plus higher order terms which we neglect, where z is the distance along the second line from the point on it closest to x. The resulting Gaussian integral with respect to z immediately gives (5.3). (Of course, calculating the effect of scalar exchange in this way with the exact propagator (5.1) gives the Yukawa potential (5.3) without any approximations at all.)

5.1.2 Factors in the ansatz

Our ansatz for the behaviour of $f_0(W_1, \ldots, W_n, g^2)$, for $n \geq 2$, when the sizes and separations of $W_1, \ldots, W_n$ are all $\geq \frac{1}{\mu}$, is the product of the following factors:
(i) a factor $e^{-\mu^2 A} = e^{-\mu^2(A_1+\ldots+A_p)}$, where A is the total area of our absolute minimal-area orientable spanning surface S of $W_1, \ldots, W_n$, and $A_1, \ldots, A_p$, $1 \leq p \leq n$, are the areas of the separate connected components $S_1, \ldots, S_p$ of S.
(ii) for each separate straight line in our minimal-length spanning tree of $S_1, \ldots, S_p$, a factor:

$$\sqrt{\frac{m}{32\pi^3 L^3}} e^{-mL} \tag{5.4}$$

where L is the length of that straight line, and $m > 0$ is the mass of the lightest glueball.
(iii) for each point where a straight line of our minimal-length spanning tree ends on one of the connected components of s, a factor f, where f represents the coupling of the lightest glueball to a minimal-area orientable spanning surface.
(iv) for each junction where three straight lines of our minimal-length spanning tree meet, a factor h, where h represents the three-glueball coupling constant.

Thus our ansatz depends on precisely four parameters, namely μ, m, f, and h. In this paper we will give the first approximation to the ratio $\frac{m}{\mu}$, leaving the first calculations of f, h, and the ratio $\frac{\mu}{\Lambda}$, where Λ, (in the range 0.1 GeV to 0.5 GeV), is the standard QCD running coupling parameter [42], to our next paper.

5.1.3 Short-distance factors

Of course in practice we have to divide $f_0(W_1, \ldots, W_n, g^2)$ by a short-distance factor for each W_i, which takes the form of the sum of all the Feynman diagrams contributing to $f_0(W_i, g^2)$, but with all the long-distance effects removed, (most simply by cutting off all the propagators smoothly at long distances), and which cancels the divergences which occur when a subdiagram of a Feynman diagram contributing to $f_0(W_1, \ldots, W_n, g^2)$, which is attached to W_i but is not connected by any *propagators* to any other part of the Feynman diagram, shrinks to a very small size on W_i [43]. We then have to rewrite the group-variation equations in terms of these ratios, and our ansatz applies to these ratios. The ratios contain a new parameter, namely the length X associated with the long-distance cutoff of the propagators in the short-distance factors, but the possible dependence on this parameter is constrained by the fact that it must cancel out of all physical quantities. In fact, if we calculate the ratios for one value, X_1, of this parameter, and if we also calculate the (finite) ratio of the *short*-distance factors for X_1 and for another value, X_2, and multiply by this ratio of *short*-distance factors, then the dependence on X_1 must cancel out, to be replaced by the equivalent dependence on X_2.

However it will not be necessary to divide by the short-distance factors in this paper, and we thus, for this paper, apply our ansatz directly to $f_0(W_1, \ldots, W_n, g^2)$. We note that the ansatz encompasses the Wilson area law for $n = 1$, and we thus apply it for all $n \geq 1$.

5.2 Substituting in the ansatz

We will see that the group-variation equations force us to make one small, but crucial, change to the ansatz, which will *not* spoil the Wilson area law or massive glueball saturation, and that the modified ansatz will then give a consistent solution of the group-variation equations when the sizes and separations of $W_1, \ldots, W_n$ are all $\geq \frac{1}{\mu}$. But first let us consider the qualitative results of substituting our ansatz into the right-hand sides of the group-variation equations.

5.2.1 *Every* 45-path now gets the effective mass

We first note that, by our ansatz, *every* 45-path in a right-hand side group-variation equation diagram, irrespective of whether the windows beside it are simply-connected or multiply-connected, now gets the same effective mass that we derived before. The most conservative, (i.e. the smallest), likely value of this effective mass is, as explained after equation (4.90), $\mu\sqrt{\frac{\pi}{2}} = 1.3\mu$. We note here that another possible cause of the logarithmic divergence in (4.88) for small q, in addition to our neglect of the effects of the ends of the paths, may be that our use, in (4.78), of a parameter along the path that is simply proportional to the number of straight line segments counted along the path from one end, rather than the parameter that truly minimizes Douglas's functional, may be all right "locally", i.e. for q not too small, but give a systematic over-estimate of the suppression, (i.e. over-estimate of the contribution to the effective mass), for the long-wavelength modes of small q. This would support our suggestion that the integrand in (4.88) be replaced, for all $0 \leq q \leq \pi$, by the value $\frac{\pi}{2}$ it takes at $q = \pi$.

5.3 The island-diagram mechanism

Let us now consider the result of substituting our ansatz into an *island* diagram in the right-hand side of the group-variation equation (4.74) for $f_0(W_1, g^2)$. Let us suppose that the loop W_1 is roughly planar, that it has no self-intersections, and that its size is large compared to $\frac{1}{\mu}$. Now the contributions of this diagram will be dominated by the contributions of islands of size roughly $\frac{1}{\mu}$, due to the effective mass that the 45-paths get, and this will be approximately true no matter how large the loop W_1 is. Furthermore, when the island is roughly in the plane of W_1, or more precisely, roughly "within" the minimal-area orientable spanning surface of W_1, (since we don't assum W_1 is *exactly* planar), and when the island is also roughly configured, in position space, such that the orientation of its "outer boundary" is roughly consistent with the orientation of the minimal-area orientable spanning surface of W_1, then by our ansatz, the window weight associated with the window that *surrounds* the island in the island diagram, namely $f_0(W_1, W_2, g^2)$, where W_2 is the closed loop in position space defined by the outer boundary of the island, will be roughly equal to $e^{-\mu^2 A_{12}}$, where A_{12} is the area of the minimal-area orientable spanning surface of W_1 and W_2. Now under the conditions stated, A_{12} will be approximately equal to A, where A is the area of

the minimal-area orientable spanning surface of W_1, minus the area of the minimal-area orientable spanning surface of W_2. But the area of the minimal-area orientable spanning surface of W_2 is proportional to $\frac{1}{\mu^2}$, and by assumption, A is large compared to $\frac{1}{\mu^2}$, hence A_{12} will be approximately equal to A, hence the factor $e^{-\mu^2 A_{12}}$ associated with the window that surrounds the island, is approximately equal to $e^{-\mu^2 A}$. And this will be true under the conditions stated, no matter where the island, of size roughly $\frac{1}{\mu}$, lies within the minimal-area orientable spanning surface of the much larger loop W_1. Hence, provided the contributions from configurations where the centre of the island does *not* lie within the minimal-area orientable spanning surface of W_1, fall off rapidly enough as the distance betweent he centre of the island, and the nearest point on the minimal-area orientable spanning surface of W_1, increases, we may expecte that the contribution of this island diagram to the right-hand side of the group-variation equation (4.74) for $f_0(W_1, g^2)$, is equal to a constant, Y, times A, the area of the minimal-area orientable spanning surface S of W_1, times $e^{-\mu^2 A}$, or in other words, is equal to $YAf_0(W_1, g^2)$, where Y is equal to the integral over all configurations of the island, subject to the projection onto S of the position of its centre, being fixed. For large W_1, and for the projection of the centre of the island onto S not being too close to the edge of S, (i.e. to W_1), we may expect Y to be roughly independent of the position of the projection of the centre of the island onto S. Furthermore, we may expect this to be the predominant behaviour of *every* island diagram, with the only difference between different island diagrams, being different values of the constant Y.

We now define L to be the "diameter" of W_1, or, in other words, L to be the largest distance between any two points of W_1. We consider a family of loops, each identical in shape to W_1, but differing in size, i.e. having a different value of L. We denote the member of this family whose diameter is L by W_{1L}, and we denote the area A_L of the minimal-area orientable spanning surface S_L of W_{1L} by $a_1 L^2$, where a_1 is of course independent of L. Then we immediately see that for large L, the contribution of each *island* diagram to the right-hadn side of the group-variation equation for $f_0(W_{1L}, g^2)$, is equal to a constant times $a_1 L^2 f_0(W_{1L}, g^2)$, while the contribution of each *non*-island diagram is not greater than a constant times $Lf_0(W_{1L}, g^2)$, (since the first non-island diagram is of course simply $f_0(W_{1L}, g^2)$, while every other non-island diagram has exactly one band, which moves like a blob of size $\frac{1}{\mu}$ along the path W_{1L}). Thus for large L, the dominant contributions to the right-hand side of the group-variation equation (4.74) for $f_0(W_{1L}, g^2)$ come from the *island* diagrams.

Let us now consider the left-hand side of the group-variation equation (4.74) for $f_0(W_{1L}, g^2)$. Now within the scope of this paper, where we do not divide the f_0's by short-distance factors, $f_0(W_{1L}, g^2)$ will satisfy the simple renormalization group equation (3.19). (The short-distance factors will give small additional terms in (3.19) which will not alter our general results, and which we neglect in this paper.) Now

$$\beta(g)\frac{\partial}{\partial g} = 2g\beta(g)\frac{\partial}{\partial g^2} \tag{5.5}$$

hence, multiplying both sides of the group-variation equation (4.74) for $f_0(W_{1L}, g^2)$ by:

$$-\frac{2g\beta(g)}{g^2} \tag{5.6}$$

we see that the term $g^2 \frac{\mathrm{d}}{\mathrm{d}g^2} f_0(W_{1L}, g^2)$ in the left-hand side of (4.74) gives simply:

$$L\frac{\partial}{\partial L} f_0(W_{1L}, g^2) \tag{5.7}$$

But by our ansatz, $f_0(W_{1L}, g^2)$ is equal to $e^{-\mu^2 a_1 L^2}$, hence (5.7) is equal to:

$$L\frac{\partial}{\partial L} e^{-\mu^2 a_1 L^2} = -2\mu^2 a_1 L^2 f_0(W_{1L}, g^2) \tag{5.8}$$

Hence this is the leading term in the left-hand side of (4.74) for large L, with the term $f_0(W_{1L}, g^2)$ being smaller by two powers of L. And comparing with our previous estimate of the right-hand side of (4.74) for large L, which has now become, due to multiplying by (5.6), the sum, over all the island diagrams, of:

$$-\frac{2g\beta(g)}{g^2} Y_i a_1 L^2 f_0(W_{1L}, g^2) \tag{5.9}$$

where the subscript i on Y_i identifies the island diagram concerned, we see that, within our estimate, our ansatz satisfies the group-variation equation (4.74) for $f_0(W_{1L}, g^2)$ for large L, *provided the sign comes out right.* And furthermore, with this proviso, μ^2 is given by:

$$\mu^2 = \frac{\beta(g)}{g} \sum_i Y_i \tag{5.10}$$

where the sum on i runs over all the island diagrams.

Now before we consider the crucial question of the sign, we note, firstly, that our estimate has in fact not been quite right. When we come to estimate more precisely the contribution of an island diagram to the right-hand side of (E184) for large L, we

find, indeed, a factor of the area, as expected, but we also find another, unwanted, factor equal to $\ln(\mu^2 A)$. This extra factor is unacceptable. It arises because the *rate* at which $f_0(W_{1L}, W_2, g^2)$ falls off, according to our ansatz, as the small loop W_2, of size roughly $\frac{1}{\mu}$, and oriented consistently with the minimal-area orientable spanning surface S_{1L} of W_{1L}, moves out of the two dimensions of S_{1L}, *decreases,* as L increases by a factor of the reciprocal of $\ln(\mu^2 A)$, rather than being independent of A. This will force us to make a small change to our ansatz, after which the Wilson area law will satisfy the group-variation equations in exactly the manner just described.

We also note here, secondly, that an island diagram mechanism, very similar to what we have just described for the area law, also results in the massive glueball saturation of the correlation functions, as the *separations* of the loops $W_1, \ldots, W_n$ increase while their *sizes* remain fixed, giving a consistent solution of the group-variation equations. In this case we will find that our ansatz works perfectly without any alterations at all.

5.4 The signs of the island diagrams, and the critical value of g^2

But before giving the details of these calculations, we now consider the crucial question of the sign of the island diagrams. We first note that the relative sign of the two terms in (3.19) is indeed correct, and in accord with the conventional form of the renormalization group equations and the conventional defintion of the β function [44]. The renormalization group equations are normally expressed in terms of a normalizaton mass μ. This normalization mass μ is *not* the same as the mass μ that occurs in the Wilson area law, notwithstanding the unfortunate use of the same symbol. Rather the *normalization* mass μ is an *input* to the QCD calculation, along with an *input* value of the coupling constant g.

5.4.1 BPHZ renormalization, and restoration of Ward identities by finite counterterms

In our BPHZ approach [45], the role of the normalization mass μ is played by $\frac{1}{R}$, where R is the length that characterizes the smooth *long*-distance cutoff that we have to impose on gluon and Fadeev-Popov propagators in counterterms, in order to avoid *long*-distance divergences in counterterms. Ward identities are restored by the addition

of simple finite counterterms [70]. For example, if $\sigma \equiv (x-y)^2$, and $b(\sigma)$ and $c(\sigma)$ satisfy

$$2\frac{\mathrm{d}}{\mathrm{d}\sigma}(b(\sigma)+\sigma c(\sigma))+3c(\sigma)=0 \tag{5.11}$$

for all $\sigma > 0$, and $b(\sigma)$ is bounded by a constant times σ^{-3} as $\sigma \to 0$ and $c(\sigma)$ is bounded by a constant times σ^{-4} as $\sigma \to 0$, then the following renormalized expression:

$$\left\{ \left\{ \int\int \mathrm{d}^4x \mathrm{d}^4y \left(\delta_{\mu\nu} b(\sigma) + (x-y)_\mu (x-y)_\nu c(\sigma)\right) \quad \times \right.\right.$$

$$\times A^{(x)}_{\mu a} \left(A^{(y)}_{\nu a} - \theta(R^2-\sigma) \left(A^{(x)}_{\nu a} + (y-x)_\alpha \partial_\alpha A^{(x)}_{\nu a} + \frac{1}{2}(y-x)_\alpha (y-x)_\beta \partial_\alpha \partial_\beta A^{(x)}_{\nu a} \right) \right) \Bigg\}$$

$$+ \quad \frac{\pi^2 R^4}{2} \left(b(R^2) + R^2 c(R^2) \right) \int \mathrm{d}^4x \left(A^{(x)}_{\mu a} A^{(x)}_{\mu a} + \frac{R^2}{12} A^{(x)}_{\mu a} \partial^2 A^{(x)}_{\mu a} \right) \Bigg\} \tag{5.12}$$

(where $\theta(s)$ is the step function, $\theta(s) = 1$ for $s \geq 0$, $\theta(s) = 0$ for $s < 0$), is finite, and is also exactly gauge-invariant under the linear gauge transformation $A_{\mu a} \to A_{\mu a} + \partial_\mu \epsilon_a$. The linear gauge variation of the additional finite counterterm, (the third line of (5.12)), exactly cancels the linear gauge variation of the standard $BPHZ$ renormalized form, (the first two lines of (5.12)). Furthermore, since (5.12) is finite and linearly gauge-invariant for *all* $R > 0$, and the R-dependent terms in (5.12) are all *local* functionals of $A_{\mu a}$, (involving at most two derivatives), the derivative of (5.12) with respect to R^2 is a finite, linearly gauge-invariant, *local* functional of $A_{\mu a}$. In fact, the derivative of (5.12) with respect to R^2, is equal to the manifestly linearly gauge-invariant expression:

$$\frac{\pi^2 R^6 c(R^2)}{24} \int \mathrm{d}^4x A^{(x)}_{\mu a} \left(\partial^2 A^{(x)}_{\mu a} - \partial_\mu \partial_\nu A^{(x)}_{\nu a} \right) \tag{5.13}$$

This is an example of the fact that, although we do indeed have, in our BPHZ approach, some power-counting renormalizable counterterms of non-gauge-invariant structure, these non-gauge-invariant counterterms have no physical effects at all, because they are completely independent of R. This is in complete contrast to the gauge-invariant counterterms, which depend on R, and have finite derivatives with respect to R^2. In our approach, the renormalization group arises from the fact that we can exactly compensate for changes of R from one finite value, strictly greater than zero, to another finite value, strictly greater than zero, by appropriate finite rescalings of g^2 and $A_{\mu a}$. We take the view that the BPHZ-renormalized perturbation expansion is generated by a "seed" action which, for the ordinary Feynman diagram expansion,

(as opposed to the group-variation equations), is the integral of the standard action density:

$$\frac{N}{4g^2}F_{\mu\nu a}F_{\mu\nu a} + \frac{N}{g^2}\left(iB_a(\partial_\mu A_{\mu a}) + \frac{\alpha}{2}B_a B_a + \psi_a(\partial_\mu(\mathrm{D}_\mu\phi)_a)\right) \tag{5.14}$$

where $F_{\mu\nu a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} + f_{abc}A_{\mu b}A_{\nu c}$ and $(\mathrm{D}_\mu\phi)_a = \partial_\mu\phi_a + A_{\mu b}f_{abc}\phi_c$. The full action is equal to the seed action plus the counterterms. The seed action depends on one physical parameter, namely g^2, (since dependence on α cancels out of physical quantities), and the counterterms depend on two physical parameters, namely g^2 and R. We define a canonical procedure, given in detail in our next paper, where at each loop order we have the canonical BPHZ counterterms, as defined in our previous paper, [26], [46], [45], plus precisely-defined Ward-identity-restoring finite counterterms, of which (5.12) is the simplest example. The canonical BPHZ counterterms depend on R through the smooth long-distance cutoffs imposed on the propagators in the counterterms. For example, if t is a fixed real number strictly greater than 1, and $f(s)$ is a member of $\mathbf{R}^{\mathbf{R}}$ that is infinitely differentiable for all $s \in \mathbf{R}$, equal to 1 for all $s \leq 1$, and equal to 1 for all $s \geq t$, we may multiply each propagator in a counterterm by $f\left(\frac{(x-y)^2}{R^2}\right)$, where x and y are the positions of the ends of the propagator. A possible form for $f(s)$ for $1 \leq s \leq t$ is $f(s) = \frac{1}{1+e^{\frac{-1}{s-1}}e^{\frac{1}{t-s}}}$. (The expression (5.12) is easily generalized to deal with a proper smooth long-distance cutoff in the counterterms rather than the simple step-function cutoff as in (5.12).)

The Ward identity is expressed in terms of the effective action Γ, (the generating functional of the proper vertices), in a standard manner [28]. Indicating position arguments by subscripts x, y, ... , and introducing sources $J_{\mu ax}$ and K_{ax} for $A_{\mu ax}$ and B_{ax}, respectively, and anticommuting sources ξ_{ax} and ζ_{ax} for ϕ_{ax} and ψ_{ax}, respectively, we define $W(J, K, \xi, \zeta)$, the generating functional of the connected Green functions, (i.e. the correlation functions), by:

$$e^W = e^{W(J,K,\xi,\zeta)} = \left\langle e^{JA+KB+\xi\phi+\zeta\psi}\right\rangle \tag{5.15}$$

where the angular brackets indicate the standard functional average over the A, B, ϕ, and ψ fields, with the weight given by the exponential of the negative of the integral of the action density (5.14). We indicate the derivative with respet to a quantity by putting the $\hat{}$ above that quantity, for example $\hat{J}_{\mu ax}$ means $\frac{\delta}{\delta J_{\mu ax}}$. Then we define the "classical fields" in the standard way by:

$$A_{\mu ax} = \hat{J}_{\mu ax}W, \qquad B_{ax} = \hat{K}_{ax}W, \qquad \phi_{ax} = \hat{\xi}_{ax}W, \qquad \psi_{ax} = \hat{\zeta}_{ax}W \tag{5.16}$$

(where these "classical fields" of course have no relation to the integration variables in the functional integrals in (5.15), notwithstanding the use of the same symbols), and we define the effective action Γ by:

$$\Gamma = JA + KB + \xi\phi + \zeta\psi - W \tag{5.17}$$

The independent variables J, K, ξ, and ζ are to be expressed in terms of A, B, ϕ, and ψ, which are then taken as the independent variables of Γ. Hence:

$$J_{\mu ax} = \hat{A}_{\mu ax}\Gamma, \qquad K_{ax} = \hat{B}_{ax}\Gamma, \qquad \xi_{ax} = -\hat{\phi}_{ax}\Gamma, \qquad \zeta_{ax} = -\hat{\psi}_{ax}\Gamma \tag{5.18}$$

We define the matrix:

$$M = \begin{pmatrix} \hat{A}_{\mu ax}\hat{A}_{\nu by}\Gamma & \hat{A}_{\mu ax}\hat{B}_{by}\Gamma & \hat{A}_{\mu ax}\hat{\phi}_{by}\Gamma & \hat{A}_{\mu ax}\hat{\psi}_{by}\Gamma \\ \hat{B}_{ax}\hat{A}_{\nu by}\Gamma & \hat{B}_{ax}\hat{B}_{by}\Gamma & \hat{B}_{ax}\hat{\phi}_{by}\Gamma & \hat{B}_{ax}\hat{\psi}_{by}\Gamma \\ \hat{\phi}_{ax}\hat{A}_{\nu by}\Gamma & \hat{\phi}_{ax}\hat{B}_{by}\Gamma & \hat{\phi}_{ax}\hat{\phi}_{by}\Gamma & \hat{\phi}_{ax}\hat{\psi}_{by}\Gamma \\ \hat{\psi}_{ax}\hat{A}_{\nu by}\Gamma & \hat{\psi}_{ax}\hat{B}_{by}\Gamma & \hat{\psi}_{ax}\hat{\phi}_{by}\Gamma & \hat{\psi}_{ax}\hat{\psi}_{by}\Gamma \end{pmatrix} \tag{5.19}$$

and its right inverse:

$$N = \begin{pmatrix} \hat{J}_{\mu ax}\hat{J}_{\nu by}W & \hat{J}_{\mu ax}\hat{K}_{by}W & \hat{J}_{\mu ax}\hat{\xi}_{by}W & \hat{J}_{\mu ax}\hat{\zeta}_{by}W \\ \hat{K}_{ax}\hat{J}_{\nu by}W & \hat{K}_{ax}\hat{K}_{by}W & \hat{K}_{ax}\hat{\xi}_{by}W & \hat{K}_{ax}\hat{\zeta}_{by}W \\ -\hat{\xi}_{ax}\hat{J}_{\nu by}W & -\hat{\xi}_{ax}\hat{K}_{by}W & -\hat{\xi}_{ax}\hat{\xi}_{by}W & -\hat{\xi}_{ax}\hat{\zeta}_{by}W \\ -\hat{\zeta}_{ax}\hat{J}_{\nu by}W & -\hat{\zeta}_{ax}\hat{K}_{by}W & -\hat{\zeta}_{ax}\hat{\xi}_{by}W & -\hat{\zeta}_{ax}\hat{\zeta}_{by}W \end{pmatrix} \tag{5.20}$$

where we note that:

$$MN = 1 \tag{5.21}$$

follows immediately from (5.16) and (5.18). (The summation convention is applied to *all* repeated indices, Lorentz, group, and position, in the matrix multiplication.) The Ward identity to be satisfied by Γ may then be written:

$$\begin{aligned} 0 = \int \mathrm{d}^4x \Big\{ & \left(\hat{A}_{\mu ax}\Gamma\right)\left(\hat{x}_\mu\phi_{ax} + A_{\mu bx}f_{abc}\phi_{cx} + f_{abc}N_{A_{\mu bx}\phi_{cx}}\right) \\ & - \frac{1}{2}f_{abc}\left(\hat{\phi}_{ax}\Gamma\right)\left(\phi_{bx}\phi_{cx} - N_{\phi_{bx}\phi_{cx}}\right) + i\left(\hat{\psi}_{ax}\Gamma\right)B_{ax}\Big\} \end{aligned} \tag{5.22}$$

It expresses the invariance of Γ under a modified BRST variation, which differs from the standard BRST variation, under which the seed action (5.16) is invariant, by the addition of the term:

$$f_{abc}N_{A_{\mu bx}\phi_{cx}} = f_{abc}\hat{J}_{\mu bx}\hat{\xi}_{cx}W \tag{5.23}$$

to the variation of $A_{\mu ax}$, and the addition of the term:

$$\frac{1}{2} f_{abc} N_{\phi_{bx}\phi_{cx}} = -\frac{1}{2} f_{abc} \hat{\xi}_{bx} \hat{\xi}_{cx} W \tag{5.24}$$

to the variation of ϕ_{ax}. These additional terms are themselves expressed in terms of Γ by (5.19) and (5.21).

The matrix N may be developed in a standard loop expansion, which, with our conventions, is equivalent to an expansion in powers of g^2. In a schematic notation, where we represent all fields by Φ, ignore Bose-Fermi distinctions, and indices i, j, ..., run over all field degrees of freedom, (i.e. over field type, and also over Lorentz, group, and position, as appropriate to each field type), and where A denotes the seed action, the first term in N, (i.e. the zero-loop term), which is of order g^2, is the matrix inverse of $\frac{\delta^2 A(\Phi)}{\delta\Phi_i \delta\Phi_j}$. In fact, if we denote $\frac{\delta^2 A(\Phi)}{\delta\Phi_i \delta\Phi_j}\Big|_{\Phi=0}$ by m_{ij}, and define n_{ij} by $m_{ik} n_{kj} = \delta_{ij}$, then the zero-loop term in N is:

$$N_{1ij} = n_{ij} - n_{ik} \left(\frac{\delta^2 A(\Phi)}{\delta\Phi_k \delta\Phi_l} - m_{kl} \right) n_{lj} + n_{ik} \left(\frac{\delta^2 A(\Phi)}{\delta\Phi_k \delta\Phi_l} - m_{kl} \right) n_{lp} \left(\frac{\delta^2 A(\Phi)}{\delta\Phi_p \delta\Phi_q} - m_{pq} \right) n_{qj} - \ldots \tag{5.25}$$

where the subscript 1 of N_{1ij} refers to the power of g^2 that N_{1ij} includes, not the number of loops. We note that n_{ij} is the matrix of free propagators, and that N_{1ij} is the corresponding matrix of propagators in the presence of the "background fields" Φ. The one-loop and higher-loop terms in N all have the form of Feynman-diagram propagator correction with the appropriate number of loops, where each line in the Feynman diagram is interpreted as N_{1ij}, each cubic vertex is interpreted as $-\frac{\delta^3 A(\Phi)}{\delta\Phi_k \delta\Phi_p \delta\Phi_q}$, and each quartic vertex is interpreted as $-\frac{\delta^4 A(\Phi)}{\delta\Phi_k \delta\Phi_l \delta\Phi_p \delta\Phi_q}$. For example, the one-loop term in N, which we denote by N_{2ij} because it includes two powers of g^2, is given by:

$$\begin{aligned} N_{2ij} = \Big\{ & \frac{1}{2} N_{1ik} \left(\frac{\delta^3 A(\Phi)}{\delta\Phi_k \delta\Phi_p \delta\Phi_q} \right) N_{1pr} N_{1qs} \left(\frac{\delta^3 A(\Phi)}{\delta\Phi_l \delta\Phi_r \delta\Phi_s} \right) N_{1lj} \\ & - \frac{1}{2} N_{1ik} \left(\frac{\delta^4 A(\Phi)}{\delta\Phi_k \delta\Phi_l \delta\Phi_p \delta\Phi_q} \right) N_{1pq} N_{1lj} \Big\} \end{aligned} \tag{5.26}$$

The counterterms are determined, for the chosen value of R, inductively in the number of loops, so that (1.24) is satisfied. At each loop order, we begin by including the canonical BPHZ counterterms, as defined in our previous paper [26], [46], [45], as generated by the seed action plus all the Ward-identity-restoring finite counterterms found at lower loop orders. These canonical BPHZ counterterms depend on R, which is by definition the largest value of $|x-y|$ for which a propagator in a counterterm,

with ends at x and y, is equal to the corresponding unmodified propagator. (In other words, R is the value of $|x-y|$ where the smooth long-distance cutoffs imposed on the propagators in the counterterms, begins to set in. The fact that the propagators in our counterterms differ from the corresponding propagators in the "direct" terms for $|x-y|>R$ was of course allowed for in all the proofs in our previous paper [26], [46], [45]. In Theorems 1 and 2 in that paper, we only required the existence of a finite real number $S>0$ such that the propagators in the counterterms are equal to the corresponding propagators in the "direct" terms for all $|x-y|\leq S$.) The inclusion of the canonical BPHZ counterterms makes Γ a finite functional of the "classical fields" $A_{\mu ax}$, B_{ax}, ϕ_{ax}, and ψ_{ax}, up to and including the current loop order, but (5.22) is not in general satisfied, (just as the first two lines of (5.12) give a finite functional of $A^{(x)}_{\mu a}$, which is not, however, linearly gauge-invariant). We then add precisely defined, R-dependent, finite counterterms, (analogous to the third line of (5.12)), after which (5.22) is exactly satisfied up to and including the current loop order. (For details we refer to our next paper.)

We then find, as we mentioned before, that if we *change* R to another finite value, say R_2, also strictly greater than zero, then this is precisely equivalent to leaving R *unaltered*, and making instead appropriate finite rescalings of g^2, α, and the fields. In fact, if we denote the seed action (5.14) by $S(A,B,\phi,\psi,g^2,\alpha)$, and the sum of all the counterterms, generated from $S(A,B,\phi,\psi,g^2,\alpha)$, by the procedure just described, by $C(A,B,\phi,\psi,g^2,\alpha,R)$, (so that the full action is equal to $S(A,B,\phi,\psi,g^2,\alpha)+C(A,B,\phi,\psi,g^2,\alpha,R)$), then we find the identity:

$$S(A,B,\phi,\psi,g^2\alpha)+C(A,B,\phi,\psi,g^2,\alpha,R_2)=$$

$$=S\left(\frac{Z_1}{Z_3}A,\frac{Z_1}{Z_3^2}B,\phi,\tilde{Z}_3\psi,\frac{Z_1^2}{Z_3^3}g^2,Z_3\alpha\right)+C\left(\frac{Z_1}{Z_3}A,\frac{Z_1}{Z_3^2}B,\phi,\tilde{Z}_3\psi,\frac{Z_1^2}{Z_3^3}g^2,Z_3\alpha,R\right) \quad (5.27)$$

where Z_3, Z_1, and $\tilde{Z}_3$ are *finite* functions of g^2, α, and $\frac{R_2}{R}$, given through order g^2, (i.e. through one-loop order), by:

$$Z_3=1+\frac{g^2}{48\pi^2}(26-6\alpha)\ln\left(\frac{R_2}{R}\right)$$

$$Z_1=1+\frac{g^2}{48\pi^2}(17-9\alpha)\ln\left(\frac{R_2}{R}\right)$$

$$\tilde{Z}_3=1+\frac{g^2}{48\pi^2}(9-3\alpha)\ln\left(\frac{R_2}{R}\right) \quad (5.28)$$

(Only the renormalization of the product $\psi\phi$ is determined - the manner in which the factor $\tilde{Z}_3$ is divided between ϕ and ψ is arbitrary.)

We note that, since the seed action is of order $\frac{1}{g^2}$, while the leading counterterms, (i.e. the one-loop counterterms), are independent of g^2, we may, correct through the terms of orders $(g^2)^{-1}$ and $(g^2)^0$ in (5.27), set all the Z factors equal to 1 in the *counterterm* term in the right-hand side of (5.27). Hence correct through this order, $S\left(\frac{Z_1}{Z_3}A, \frac{Z_1}{Z_3^2}B, \phi, \tilde{Z}_3\psi, \frac{Z_1^2}{Z_3^3}g^2, Z_3\alpha\right) - S(A, B, \phi, \psi, g^2, \alpha)$ is simply equal to $C(A, B, \phi, \psi, g^2, \alpha, R_2) - C(A, B, \phi, \psi, g^2, \alpha, R)$, which is how (5.28) is obtained.

5.4.2 The critical value of g^2

Now as we mentioned above in the discussion following the statement of our ansatz, we in practice have to divide $f_0(W_1, \ldots, W_n, g^2)$ by a short-distance factor for each W_i, which takes the form of the sum of all the Feynman diagrams contributing to $f_0(W_i, g^2)$, but with all the propagators cut off smoothly at long distances, in the same manner as in the counterterms, but with the onset of the long-distance cutoffs possibly occurring at a different value of $|x - y|$, say at $|x - y| = T$, (where T is finite and strictly greater than zero), from the onset value R of the long-distance cutoffs on the propagators in the counterterms. We allow T to differ from R because, as we will see, the real number $\frac{1}{\mu R}$, where μ^2 is the coefficient of the area in the Wilson area law in our ansatz, is computable from the group-variation equations in terms of the "input" value of g^2, as the value of $\frac{L}{R}$ where the running coupling $\bar{g}^2\left(\frac{L}{R}\right)$, with the initial value $\bar{g}^2(1) = g^2$, reaches a critical value that is computable from the group-variation equations. The critical value is not a fixed point: it is the point where the magnitude of $\frac{\beta(g)}{g}$ reaches a sufficiently *large* value, as can be seen from (5.10).

5.4.3 The signs of the island diagrams

The values of Y_i for the three one-loop island diagrams have no explicit dependence on g^2. Obviously, $\sum_i Y_i$ has got to be *negative*, and, indeed, the net contribution to $\sum_i Y_i$ from the three one-loop island diagrams has surely also got to be *negative*. We will return to this crucial question shortly.

5.4.4 g^2 never evolves to a value larger than the critical value

We note here that, once g^2 reaches the critical value, the renormalization group has done its job. The right-hand side of the group-variaton equation for $f_0(W_1, \ldots, W_n, g^2)$, whenever any of the sizes or separations of the W_i's is larger than $\frac{1}{\mu}$, is dominated by the contributions of island diagrams of size $\frac{1}{\mu}$. Thus no matter how large the sizes and separations of the W_i's, the running coupling never evolves to a scale larger than $\frac{1}{\mu}$, and never reaches a size larger than the critical value.[1] This is in agreement with experiment: there is no concrete experimental evidence that α_s ever reaches a value significantly larger than the value 0.3 found for charmonium [47]. Note added: the largest value of α_s for which there is currently experimental evidence, is the value $\alpha_s(m_\tau) = \alpha_s(1748 \text{ MeV}) = 0.35$, observed in τ decay [5]. As mentioned in the Introduction, it is interesting to note that in Figure 9.2, on page 19, of Chapter 9, *Quantum Chromodynamics*, of reference [5], the experimental value of $\alpha_s(m_\tau)$ lies approximately 1.6 standard deviations above the best-fit curve to all measurements of α_s, which indicates that the curve of $\alpha_s(\mu)$ may be curving upwards towards a vertical slope here, as expected as α_s approaches the critical value. In fact, as discussed in the Introduction, and in Subsection 6.2.1, the large-N_c limit, of the general form of the four-loop β-function given in reference [9], indicates that the series for our $\frac{\beta(g)}{g}$ probably diverges for $\frac{g^2}{4\pi}$ somewhere in the range 0.53 to 1.05, and most likely near the lower end of this range. Since this $\frac{g^2}{4\pi}$ is equal to $\frac{3}{2}$ times the value α_s would have in the absence of quarks, which is about $\frac{3}{2} \times \frac{1}{1.22} \times \alpha_s = 1.23\alpha_s$ at 1748 MeV, (where the factor 1.22 corresponds to three quarks lighter than 1748 MeV), we expect the critical value of α_s to lie somewhere in the range 0.43 to 0.85, and most likely near the lower end of this range. Conversely, the observation of $\alpha_s(1748 \text{ MeV}) = 0.35$ indicates that the critical value of our $\frac{g^2}{4\pi}$ is larger than, and probably quite close to, 0.43. We should also note that $\frac{\beta(g)}{g}$ is expected to *diverge* for our $\frac{g^2}{4\pi}$ somewhere in the range 0.53 to 1.05, and the critical value of $\frac{g^2}{4\pi}$ is expected to be strictly *less* than the value of $\frac{g^2}{4\pi}$ where the series diverges, with a corresponding reduction in the expected critical value of α_s.

[1] Note added: Of course, this result depends crucially on the fact that an island diagram contains *no* propagators that are not part of the island.

5.4.5 Normalization of g^2

Our $\frac{g^2}{4\pi}$ is larger than α_s as conventionally defined by a factor $\frac{3}{2}$. The factor of 3 is due ot the explicit factor of N in (1.51) and in (5.14), while the factor of $\frac{1}{2}$ is because we normalize our fundmental representation generators by (1.44), whereas the generators traditionally used in extracting α_s from experiments are defined with an extra factor of $\frac{1}{2}$ in the right-hand side of (1.44) [48], [49], [50], [51]. Including an extra factor of $\frac{1}{2}$ in the right-hand side of (1.44) has the effect of reducing structure constants by a factor $\frac{1}{\sqrt{2}}$, hence of increasing the value of g, as extracted from any experimental result, by a factor $\sqrt{2}$, since only the product gf_{abc} is actually measured.

5.4.6 The renormalization group, and the critical value of g^2

Now in our BPHZ approach, there is no "bare" coupling constant, and we never do any infinite rescalings of the fields. Infinite rescalings of the fields would be completely pointless, because the counterterms do *not* have a gauge-invariant or BRST-invariant structure. Only the seed action has a BRST-invariant structure, and the property of the counterterms that "corresponds" to BRST-invariance is expressed by (5.27): leaving the fields, g^2, and α unaltered, and replacing R by R_2, is equivalent to leaving R unaltered, and making finite rescalings of the fields, g^2, and α, as in the right-hand side of (5.27). The seed action in the right-hand side of (5.27) also has a BRST invariance, which differs from that of the seed in the left-hand side by the rescalings of the fields.

Consequently we must consider carefully the definition and significance of $\beta(g)$. Now in conventional formulations of renormalization group equations [35], the *normalization* mass μ, which is an *input*, and corresponds to $\frac{1}{R}$ in our approach, and $\beta(g)$ always occur in the combination:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \text{ other terms }\right) f = 0 \tag{5.29}$$

Here the "other terms" depend on the identity of f, and when f is $f_0(W_1, \ldots, W_n, g^2)$, the only "other terms" are those associated with the short-distance factors, which we omit in this paper. We note that if $W_1, \ldots, W_n$ is a set of loops whose "diameter", i.e. the largest distance between any two points on the loops, is equal to $\frac{1}{\mu}$, and $W_{1L}, \ldots, W_{nL}$ denotes the set of loops obtained from $W_1, \ldots, W_n$ by uniformly rescaling all their sizes and separations by a factor μL, so that their diameter becomes L, then

$f_0(W_{1L}, \ldots, W_{nL}, g^2)$ can only depend on μ and L through the combination μL, hence the relative sign of the two terms in (3.19) is indeed correct.

Now, as we mentioned, the conventional "normalization mass" μ corresponds to our $\frac{1}{R}$. Therefore our version of (5.29) is:

$$\left(-R\frac{\partial}{\partial R} + \beta(g)\frac{\partial}{\partial g} + \text{ other terms }\right) f = 0 \tag{5.30}$$

Now when the "other terms" are absent, the meaning of (5.30) is that it gives the curves of constant f in the (R, g) plane. Indeed, under variations of our input parameters R and g, and assuming that f is a physical quantitiy, and hence independent of α:

$$\mathrm{d}f = \frac{\partial f}{\partial R}\mathrm{d}R + \frac{\partial f}{\partial g}\mathrm{d}g \tag{5.31}$$

Hence when the "other terms" are absent, the meaning of (5.30) is that the curves of constant f, i.e. $\mathrm{d}f = 0$, in the (R, g) plane, are given by:

$$R\frac{\mathrm{d}g}{\mathrm{d}R} = -\beta(g) \tag{5.32}$$

This is the physical significance of the renormalization group from our point of view. Now from equation (5.27) we see that, up to the effects of rescaling the fields and α, (which are what can give the "other terms" in (5.30)), calculated quantities f will be left unaltered if we *simultaneously* replace R by R_2 and g^2 by g_2^2, where:

$$g_2^2 = \frac{Z_3^3}{Z_1^2}g^2 \tag{5.33}$$

Hence from (5.32) we see that $\beta(g)$ is given by:

$$\beta(g) = -R_2\frac{\mathrm{d}g_2}{\mathrm{d}R_2} \tag{5.34}$$

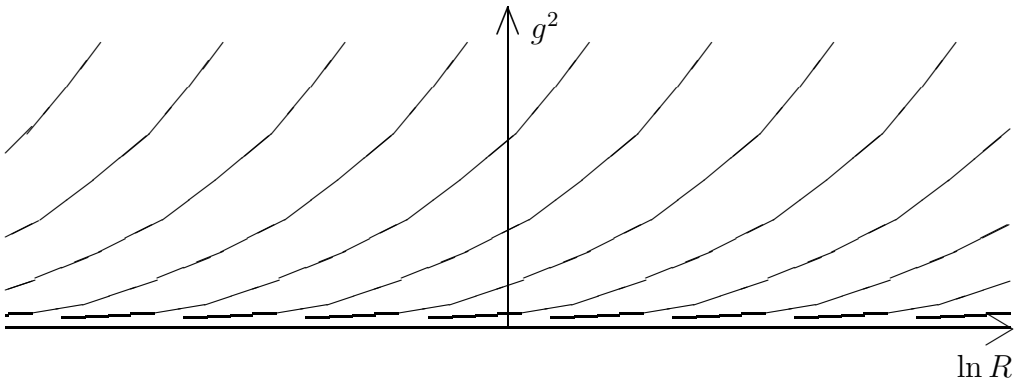
where the derivative is to be evaluated at constant R and g, after which R_2 is to be set equal to R, and g_2 set equal to g. Thus from (5.28) and (5.33) we find:

$$2g\beta(g) = -\frac{11g^4}{12\pi^2} + \text{ terms of order } g^6 \tag{5.35}$$

Thus for small g^2 the "curves of constant f" in the (R, g) plane have the form:

$$\frac{g^2}{4\pi} = \frac{1}{\frac{11}{6\pi}\ln\left(\frac{\text{constant}}{R^2}\right)} \tag{5.36}$$

The "curves of constant f" in the (R, g) plane look like:



(5.37)

Each of these curves corresponds to a different choice of the constant in (5.36), and each of these curves gives a possible behaviour of the "running coupling" $\bar{g}^2(R)$.

We note that, for the reasons given above, for each value of the logarithm in (5.36), our $\frac{g^2}{4\pi}$ is larger than α_s for the corresponding value of the logarithm [42], [52] by a factor $\frac{3}{2}$.

Now the critical value of g^2 is certainly calculable from the group-variation equations. In brief, each Y_i in the right-hand side of (5.10) will be equal, on dimensional grounds, to μ^2, tims a calculable *numerical* factor, times a power of g^2 that can be read off the corresponding island diagram. For the one-loop islands, this power of g^2 is *zero*. Thus μ^2 cancels out, and we are left with an equation of the form $1 = \frac{\beta(g)}{g}$ times a function of g^2 that begins with a term independent of g^2. Hence (5.10) fixes g^2, and the condition is roughly that $\frac{\beta(g)}{g}$ be equal to the reciprocal of the sum of the numerical factors in the Y_i corresponding to the three one-loop island diagrams. (This of course means that, as we noted, the sum of these three Y_i must be *negative*.) Now (5.10) of course depends on our ansatz, and we thus see that, as we noted above, throughout the domain described by our ansatz, (i.e. when the sizes and separations of the loops are all larger than $\frac{1}{\mu}$), g^2 remains equal to the critical value: g^2 never goes above the critical value. Now at smaller scales, the behaviour described by our ansatz has not yet been achieved, and g^2 can be smaller. What this means in practice is that we can do our calculations with an input value of g^2 as small as we like: the only restrictions on the input value of g^2 are that it must be strictly greater than zero, and not greater than the critical value. However, the input value of g^2 chosen fixes $\frac{1}{\mu R}$, (where μ^2 is the coefficient of the area in the Wilson area law in our ansatz, so that $\frac{1}{\mu}$ is the typical island size), according to the renormalization group. If we choose the input value of g^2 equal to the critical value, then $\frac{1}{\mu R}$ is a calculable number of order 1. And as we reduce

the input value of g^2 below the critical value, $\frac{1}{\mu R}$ increases very rapidly, in a manner given precisely by the renormalization group. For example, (referring to experimental data), the experimental value of μ is 0.41 GeV, as we noted before. The experimental value of α_s at about 4 GeV is approximately 0.3, [47], and the experimental value of α_s at about 40 GeV is approximately 0.16, [53]. Now our $\frac{g^2}{4\pi}$ is $\frac{3}{2}$ times the value α_s would have in the absence of quarks, and the presence of the quarks rougly increases α_s, in comparison to the value it would have in the absence of the quarks, by a factor 1.3 at 4 GeV, and by a factor 1.4 at 40 GeV. Hence we infer that the critical value of our $\frac{g^2}{4\pi}$ is greater than 0.35, and that if we choose our input value of $\frac{g^2}{4\pi}$ equal to 0.35, we will find that $\frac{1}{\mu R}$ is approximately equal to 10, (which would mean that our input parameter R is approximately equal to 0.25 GeV^{-1}), and that if we choose our input value of $\frac{g^2}{4\pi}$ equal to 0.17, we will find that $\frac{1}{\mu R}$ is approximately equal to 100, (which would mean that our input parameter R is approximately equal to 0.025 GeV^{-1}).

5.5 The result of dividing by the short-distance factors

The reason for this is as follows. Our $f_0(W_1, \ldots, W_n, g^2)$ satisfy a version of (5.30) where the "other terms" are associated exclusively with the short-distance factors which we have to divide out. These short-distance factors are characterized by a parameter T, which, as we explained above, is the value of $|x - y|$ where the long-distance cutoffs imposed on the propagators in the short-distance factors, begins. T is finite and strictly greater than zero. The value of T cancels out of all physical quantities, such as μ, and the mass m of the lightest glueball. We allow T to differ from R because, as we just noted, if we choose our input value of g^2 equal to the critical value, we will find that $\frac{1}{\mu R}$ is equal to a calculable number of order 1. Thus if we choose our input value of g^2 equal to the critical value, our input parameter R is approximately equal to the typical island size. Hence if we required T to be equal to R, we would have practical problems when we explicitly restore the short-distance factors in the right-hand sides of the group-variation equations, as we have to do, because the short-distance factors would then become subject to strong-coupling effects. Hence we allow T to differ from R, so that if we choose our input value of g^2 close to the critical value, we are free to choose T to be substantially smaller than R, to ensure that the short-distance factors are not

subject to strong-coupling effects. When we express the group-variation equations in terms of the f_0's with the short-distance factors divided out, we have to restore the short-distance factors explicitly in each window of each right-hand side group-variation equation diagram. This results in partly recovering the *Feynman* diagram expansions of the left-hand sides of the group-variation equations, but in such a way that all the long-distance information is contained in the "long-distance factors", (i.e. the f_0's of the right-hand side windows with their short-distance factors divided out), and the propagators in the "Feynman diagrams" are all cut off smoothly at long distances, with the onset of the cutoffs occurring at $|x-y| = T$. We also divide each right-hand side group-variation equation diagram by the short-distance factor associated with the left-hand side f_0, which for the island diagrams trivially cancels the restoration of the short-distance factors associated with the left-hand side loops $W_1, \ldots, W_n$, and for non-island diagrams cancels the main part of the restoration of the short-distance factors for the paths that form part of the left-hand side loops, with some complications occurring at the junctions where a 45-path ends on a left-hand side loop.

The sum of "Feynman diagrams" we obtain when we restore the short-distance factors in the windows of the right-hand side group-variation equation diagrams also differs from the Feynman diagram expansion of the left-hand side of the group-variation equation, (which, as we mentioned after equation (4.74), is simply the sum of the Feynman diagrams contributing to the left-hand side f_0, with each of these Feynman diagrams being multiplied by it number of windows, for the following reason:

We define a "band" of a Feynman diagram contributing to an expectation value or correlation function of Wilson loops, to be a connected component of what remains of the diagram whenall the Wilson loops are removed. When we substitute the short-distance factor, as defined above, into a *non-simply-connected* window of a right-hand side group-variation diagram, we recover not the perturbative expansion, with modified propagators, of the correlation function of Wilson loops that border that window, but rather the perturbative expansion, with modified propagators, of the product of the vacuum expectation values of the Wilson loops that border that window. (All factors of N have of course been removed - what we have is $\prod_i f_0(W_i, g^2)$, where the W_i's are the Wilson loops that border that window.) The Feynman diagrams contributing to this product of vacuum expectation values have no bands with ends on two or more different W_i's, in complete contrast to the Feynman diagrams contributing to the correlation function of the W_i's. What this means in practice is that it is our

"long-distance factors", (i.e. our f_0's divided by their short-distance factors, as defined above), that become singular when two *different* W_i's intersect one another. This is not a serious problem, because configurations where two different Wilson loops intersect one another are not very important in four dimensions, but it does mean that, for the full recovery of the renormalized perturbation expansion of the left-hand sides of the group-variation equations, our ansatz for the long-distance factors must be refined to include this feature.

The perturbative expansions, (i.e. in powers of g^2), of the long-distance factors, (i.e. the f_0's divided by their short-distance factors), have a simple form: they can be expressed by the same sum of Feynman diagrams that contributes to f_0, provided we modify the mathematical expression that corresponds to each diagram as follows, and also partly drop the requirement of planarity, as follows. We call the bands, (as defined above), of a Feynman diagram contributing to $f_0(W_1, \ldots, W_n)$, "mono-bands", "duo-bands", "trio-bands", and so on, according to the number of different W_i's they have ends on. Thus a mono-band has all its ends on a single W_i, a duo-band has ends on precisely two different W_i's, a trio-band has ends on precisely three different W_i's, and so on. We then relax the requirement of planarity as follows: every band is still required to be planar "within itself", and the relations among all bands other than mono-bands are still required to be planar, as before. However, the requirement of planarity in the relations between mono-bands and other bands is dropped: it is as if, to each mono-band, every other band is invisible. We define the contribution to $f_0(W_1, \ldots, W_n)$ of each band configuration that is now allowed, but which was not allowed before, to be zero. Now our long-distance factor is, by definition:

$$\frac{f_0(W_1, \ldots, W_n, g^2)}{\prod_{i=1}^{n} \tilde{f}_0(W_i, g^2)} \tag{5.38}$$

where $\tilde{f}_0(W_i, g^2)$ is defined by the same sum of Feynman diagrams as $f_0(W_i, g^2)$, but with all propagators cut off smoothly at long distances, with the onsets of the long-distance cutoffs occurring at $|x - y| = T$. We expand each factor $\frac{1}{\tilde{f}_0(W_i, g^2)}$ as:

$$\frac{1}{\tilde{f}_0(W_i, g^2)} = 1 - \Big(\tilde{f}_0(W_i, g^2) - 1\Big) + \Big(\tilde{f}_0(W_i, g^2) - 1\Big)^2 - \ldots \tag{5.39}$$

Let us now consider one of our generalized diagrams contributing to $f_0(W_1, \ldots, W_i, g^2)$, (i.e. where we have dropped the requirement of planarity with

reference to the relations between different mono-bands, and between mono-bands and other bands, and defined the contributions to $f_0(W_1, \ldots, W_n, g^2)$ from the new configurations that were previously forbidden, to be zero). Let us consider a specific configuration, in position space, of the vertices of this diagram, and let us consider *all* the contributions to (5.38) of the precise set of bands and configuration of their vertices that we have here, when we expand all the denominators in (5.38) as in (5.39). Let A represent all the bands *except* the mono-bands, i.e. A contains all the duo-bands, all the trio-bands, etc. We treat A as a single indivisible unit, since no band in A can come from anywhere except $f_0(W_1, \ldots, W_n, g^2)$, (i.e. no band in A can come from any of the $\Big(\tilde{f}_0(W_i, g^2) - 1\Big)$'s). Let the mono-bands of this generalized diagram "contributing" to $f_0(W_1, \ldots, W_n, g^2)$ be $B_1, B_2, \ldots, B_r$. We are considering a specific configuration in position space of the vertices of all the bands, hence we may assum that all B_i are different, since even if two of the B_i's look identical *as diagrams*, configurations where all their corresponding vertices have identical positions in configuration space have measure zero, and may hence be ignored. Now each mono-band B_j is attached to a specific one of the W_i's, and if B_j is attached to W_i, then B_j can also occur in $\tilde{f}_0(W_i, g^2)$, in which case we denote it by $\tilde{B}_j$, to signify that all its propagators have long-distance cutoffs, commencing at $|x - y| = T$.

Now let us suppose, for example, that there are just two mono-bands, B_1 and B_2. We then write:

$$f_0(W_1, \ldots, W_n, g^2) = \Big\{1 + \ldots + \langle A \rangle + \ldots + \langle B_1 \rangle + \ldots + \langle B_2 \rangle + \ldots + \langle AB_1 \rangle + \\ + \ldots + \langle AB_2 \rangle + \ldots + \langle B_1 B_2 \rangle + \ldots + \langle AB_1 B_2 \rangle + \ldots \Big\} \tag{5.40}$$

where the bracket notation displays all the bands in a diagram, A is an abbreviation of $D_1 \ldots D_d T_1 \ldots T_t \ldots$, where the D_i's are the duo-bands, the T_i's are the trio-bands, and so on, and each B_i, D_j, T_k, etc., indicates not only a specific band *as a diagram*, but also a specific position-space configuration of the vertices of that band, and we explicitly display all the contributions to $f_0(W_1, \ldots, W_n, g^2)$ that can contribute to the contribution to our long-distance factor (5.38) that involves *precisely* the bands in A, and also all the B_i's, where each B_i may or may not have a "twiddle".

And in the same bracket notation, if the mono-bands attached to W_i, for example, are precisely B_j and B_k, we write:

$$\tilde{f}_0(W_i, g^2) = 1 + \ldots + \Big\langle \tilde{B}_j \Big\rangle + \ldots + \Big\langle \tilde{B}_k \Big\rangle + \ldots + \Big\langle \tilde{B}_j \tilde{B}_k \Big\rangle + \ldots \tag{5.41}$$

We then expand each $\frac{1}{f_0(W_i,g^2)}$ as in (5.39), and extract all the terms that involve precisely A, and also each B_i precisely once, where each B_i may occur either without a "twiddle" or with a "twiddle", (but not both ways in the same term).

The general result is that we have a sum over all the partitions of $\langle AB_1B_2\dots B_r\rangle$ into nonempty parts, (with A being treated as an indivisible unit), such that every B_i in the *same* part as A, occurs *without* a "twiddle", and every B_i *not* in the same part as A, occurs with a "twiddle", (i.e. is $\tilde{B}_i$), and the partitions are restricted by the requirement that no two $\tilde{B}_j$'s attached to two *different* W_i's, may occur in the same part, (which means that every part that does *not* contain A, is associated with a *specific* W_i). And if, for each $1 \leq i \leq n$, the number of parts of the partition associated with W_i, (in the sense that all the $\tilde{B}_i$'s in that part, are attached to W_i), is m_i, then the coefficient of that partition is:

$$(-1)^{m_1+m_2+\dots+m_n} m_1!m_2!\dots m_n! \tag{5.42}$$

For example, if the mono-bands are B_1 and B_2, and B_1 and B_2 are both attached to the *same* W_i, we have:

$$\langle AB_1B_2\rangle - \langle AB_1\rangle\left\langle\tilde{B}_2\right\rangle - \langle AB_2\rangle\left\langle\tilde{B}_1\right\rangle - \langle A\rangle\left\langle\tilde{B}_1\tilde{B}_2\right\rangle + 2\langle A\rangle\left\langle\tilde{B}_1\right\rangle\left\langle\tilde{B}_2\right\rangle \tag{5.43}$$

while if the mono-bands are B_1 and B_2, and B_1 and B_2 are attached to two *different* W_i's, then we have:

$$\langle AB_1B_2\rangle - \langle AB_1\rangle\left\langle\tilde{B}_2\right\rangle - \langle AB_2\rangle\left\langle\tilde{B}_1\right\rangle + \langle A\rangle\left\langle\tilde{B}_1\right\rangle\left\langle\tilde{B}_2\right\rangle \tag{5.44}$$

In these formulae, and in the general result, if there are *no* bands in A, (i.e. if all the bands are mono-bands), we simply set $A = 1$. This is the form of the result that applies for $n = 1$. (of course, for $n \geq 2$, there is always at least one band in A, since every Feynman diagram contributing to $f_0(W_1,\dots,W_n,g^2)$, is connected.)

We note that if *all* the mono-bands $B_1, B_2, \dots, B_r$ are attached to the *same* W_i, then we simply have the "correlation function" of A and all the B_i's, with each B_i that is *not* in the same part as A, becoming a $\tilde{B}_i$. We also note that our general result here has nothing to do with planar diagrams or the large-N limit. In fact, the general result is more simply stated for the general-N case. The interpretation of the bracket notation is simply that for each "$\tilde{B}$ part" that is associated with a loop W_i, we have a separate "duplicate" of the loop W_i, "superimposed" on the loop W_i, such that the fundamental representation matrices t_a at the ends of the mono-bands in that "$\tilde{B}$

part", have their own matrix product and trace around that "duplicate" of the loop W_i, (which comes from the expansion of one of the denominator factors), rather than "mixing in" with the matrix products around the "original" loop W_i, which is included in the part that contains A, and comes from the numerator in (5.38).

We can now see how dividing out the short-distance factors has cancelled all the linear short-distance divergences along the loops. These occur when a mono-band, or more generally, a group, or cluster, of monobands, all on the same W_i, which may be nested, (in the large-N case, or, in the case of unrestricted N, generally "entangled" with one another, all shrink togethr to a point on the loop W_i, such that the freedom of movement of this point along the loop W_i is not restricted by this group of bands being entangled with any other band. Now when such a group of mono-bands is not entangled with any other band, (by which we mean, that the matrices t_a at the vertices where gluon propagators of these bands end on the loop W_i, occur in a single sequence, unbroken by the presence of any t_a belonging to any band *not* in this group), the structure constants f_{abc} at the action vertices in these bands, and the matrices $(t_a)_{jk}$ at the vertices where gluon propagators of these bands end on the loop W_i, are contracted into an SU(N)-invariant tensor X_{jk}, which has precisely one "quark representation index", j, one "anti-quark representation index", k, and *no* adjoint representation indices, and hence satisfies, for any infinitesimal parameters ϵ_a, $a \in \mathrm{SU}(N)$:

$$\left(\delta_{jp} + (\epsilon_a t_a)_{jp}\right) \left(\delta_{kq} + \epsilon_b \left((t_b)_{kq}\right)^*\right) X_{pq} = X_{jk} \tag{5.45}$$

which implies immediately that, for all t_a:

$$(t_a)_{jp} X_{pk} - X_{jq} (t_a)_{qk} = 0 \tag{5.46}$$

or in other words that, considered as a matrix, X_{jk} commutes with all the t_a's [43]. (Equation (5.45) follows immediately from the invariant tensor properties of f_{abc} and $(t_a)_{jk}$.) Now the t_a's are the generators of the fundamental representation of SU(N), which is irreducible, hence the fact that X_{jk} commutes with all the t_a's implies immediately that X_{jk} is a multiple of the unit matrix, $X_{jk} = X\delta_{jk}$, say.

Let us now number our bands such that the s mono-bands which are going to shrink to a point on one of the W_i's, are B_1, B_2, ..., B_s. We assume $s \geq 1$. Now all the s "shrinking" mono-bands are of course attached to a *single* one of the W_i's: let the W_i to which the s "shrinking" bands are attached, be W_1.

Let us now consider all the terms in our long-distance factor, as derived above, which involve precisely a specified set, A, of duo-bands, trio-bands, and so on, precisely the s "shrinking" mono-bands $B_1, B_2, \ldots, B_s$, and a precise set $B_{s+1}, \ldots, B_t$ of additional monobands. These terms correspond, as shown above, to all partitions of $\langle AB_1B_2 \ldots B_sB_{s+1} \ldots B_t\rangle$, such that in every part, (i.e. every bracket), that does *not* contain A, all the B_j's in that part are attached to the *same* W_i. Now bearing in mind the origins of the brackets, we call the one part of each such partition that contains A, the "numerator part", and all the other parts of each such partition, "denominator parts". Thus each denominator part is associated with a specific W_i, in the sense that all the mono-bands in that part, are attached to that W_i. Furthermore, all the B's in the numerator part are *without* "twiddles", while all the B's in every denominator part, have "twiddles". And if, for each $1 \leq i \leq n$, the number of denominator parts associated with W_i, is m_i, then the coefficient of that partition is given by (5.42).

Now the "shrinking" mono-bands, B_1, B_2, ..., B_s, are by assumption not "entangled" with any other band, in the sense defined above. Hence by the result just given, we have, for each denominator part associated to W_1, in any of our partitions, that if, for example, that denominator part is $\left\langle \tilde{B}_2\tilde{B}_5\tilde{B}_7\tilde{B}_9\tilde{B}_{15}\right\rangle$, where $\tilde{B}_2$, $\tilde{B}_5$, and $\tilde{B}_7$ are "shrinking" mono-bands, and $\tilde{B}_9$ and $\tilde{B}_{15}$ are "non-shrinking" mono-bands attached to W_1, the following identity, in our bracket notation:

$$\left\langle \tilde{B}_2\tilde{B}_5\tilde{B}_7\tilde{B}_9\tilde{B}_{15}\right\rangle = \left\langle \tilde{B}_2\tilde{B}_5\tilde{B}_7\right\rangle \left\langle \tilde{B}_9\tilde{B}_{15}\right\rangle \tag{5.47}$$

In other words, each denominator part associated to W_1, factorizes exactly into a product of two factors, one of which is the bracket containing all the "shrinking" mono-bands in that denominator part, and the other of which is the bracket containing all the "non-shrinking" mono-bands in that denominator part. (Any bracket that contains *no* bands, is by definition equal to 1. This applies, in particular, for $n = 1$, to the numerator bracket, when the numerator bracket contains just A, since for $n = 1$, there are *no* duo-bands, trio-bands, etc.)

And we also have a corresponding result for the numerator bracket: if, for example, the numerator bracket is $\langle AB_3B_6B_{10}B_{12}B_{19}\rangle$, where B_3 and B_6 are "shrinking" mono-bands, and B_{10}, B_{12}, and B_{19} are "non-shrinking" mono-bands, then we have the identity:

$$\langle AB_3B_6B_{10}B_{12}B_{19}\rangle = \langle AB_{10}B_{12}B_{19}\rangle \langle B_3B_6\rangle \tag{5.48}$$

In other words, the numerator bracket exactly factors into a product of two factors,

one of which is the numerator bracket containing just A and all the *non-shrinking* mono-bands in the original numerator bracket, and the other of which is a bracket containing all the *shrinking* mono-bands in the original numerator bracket:this factor is like a denominator bracket, but without the "twiddles".

Let us now suppose that the position-space configuration of the vertices of the "shrinking" mono-bands is sufficiently "shrunk" that , for every propagator in each of these bands, the distance between the ends of that propagator is less than or equal to T, where T is the onset point of the smooth long-distance cutoffs imposed on the propagators in the $\tilde{B}$'s. then the smooth long-distance cutoffs imposed on the "non-counterterm" propagators in the $\tilde{B}$'s have *no effect*, hence for $1 \leq j \leq s$, (where s is the number of "shrinking" mono-bands), we may set $\tilde{B}_j = B_j$. (We note that, in the *counterterms* in the $\tilde{B}$'s, we must use the same smooth long-distance cutoffs as in the counterterms *in the B's*, i.e. with the onset of the smooth long-distance cutoffs occurring at R, *not* at T.)

We thus see that, due firstly to the factorization identities (5.47) and (5.48), and secondly to the smallness, in configuration space, of the shrinking mono-bands, our original set of partitions of $\langle AB_1B_2 \dots B_sB_{s+1} \dots B_t \rangle$, has collapsed to a strict subset of this original set of partitions, whose members may be characterized as follows: a member of our "final set" of partitions of $\langle AB_1B_2 \dots B_sB_{s+1} \dots B_t \rangle$, consists of an allowed partition of $\langle AB_{s+1} \dots B_t \rangle$, together with an arbitrary partition of $\langle B_1B_2 \dots B_s \rangle$. We have to calculate the total coefficient with which we obtain each member of this "final set" of partitions. Let us consider a member of this "final set" of partitions, which has, as before, for each $1 \leq i \leq n$, a total of m_i denominator parts associated with W_i. (By a "denominator part", we still mean any part that does not contain A.) For convenience, we also define $m = m_1$. Let the number of parts of our partition of $\langle B_1B_2 \dots B_s \rangle$ be u. Thus we have $1 \leq u \leq m$. We have to identify, among all the original partitions, all the possible "sources" of this final partition. These are given by all possible mergings of k parts of our partition of $\langle B_1B_2 \dots B_s \rangle$, where $0 \leq k \leq \min(u, m+1-u)$, with either denominator parts, *associated with* W_1, of our partition of $\langle AB_{s+1} \dots B_t \rangle$, or with the numerator part of our partition of $\langle AB_{s+1} \dots B_t \rangle$, subject to the restriction that at most one part of our partition of $\langle B_1B_2 \dots B_s \rangle$, can merge with any given part of our partition of $\langle AB_{s+1} \dots B_t \rangle$. Now for each possible such merging of k parts of our partition of $\langle B_1B_2 \dots B_s \rangle$, into parts of our partition of $\langle AB_{s+1} \dots B_t \rangle$, subject to the restrictions stated, the coefficient of the original partition obtained by that merging, is

obtained from the coefficient of our "final" partition, by multiplying by:

$$(-1)^k \frac{(m-k)!}{m!} \tag{5.49}$$

since the total number of denominator parts associated with W_1 has been reduced by k. And the number of distinct possible such mergings, involoving k parts of our partition of $\langle B_1 B_2 \dots B_s \rangle$, is:

$$\frac{(m+1-u)!}{(m+1-u-k)!k!} \frac{u!}{(u-k)!k!} k! \tag{5.50}$$

Hence the total coefficient of our final partition is:

$$\frac{(m+1-u)!u!}{m!} \sum_{k=0}^{\min(u,m+1-u)} (-1)^k \frac{(m-k)!}{(m+1-u-k)!(u-k)!k!} \tag{5.51}$$

times the original coefficient of our final partition. Now, defining $v = m+1-u$, so that we have both $u \geq 1$ and $v \geq 1$, the sum in (5.51) is equal to:

$$F(u,v) = \sum_{k=0}^{\min(u,v)} (-1)^k \frac{(u+v-k-1)!}{(u-k)!(v-k)!k!} \tag{5.52}$$

This expression $F(u,v)$ is symmetric in u and v, and is well-known to vanish for all integer u and v such that $u \geq 1$ and $v \geq 1$ are both true. Indeed, we see immediately that $F(u,1) = 0$ for all integers $u \geq 1$. And for all integers u, v, such that $u \geq v \geq 2$, we have:

$$F(u,v) = \sum_{k=0}^{v} (-1)^k \frac{(u+v-k-1)!}{(u-k)!} \frac{1}{v} \left(\frac{1}{(v-k-1)!k!} + \frac{1}{(v-k)!(k-1)!} \right) =$$

$$= \frac{1}{v} \left(\sum_{k=0}^{v-1} (-1)^k \frac{(u+v-k-1)!}{(u-k)!(v-k-1)!k!} + \sum_{k=0}^{v-1} (-1)^{k+1} \frac{(u+v-k-2)!}{(u-k-1)!(v-k-1)!k!} \right) =$$

$$= -\left(\frac{v-1}{v} \right) F(u,v-1) \tag{5.53}$$

Hence by induction on v, $F(u,v) = 0$ for all integers u and v such that $u \geq v \geq 1$, hence, by symmetry, $F(u,v) = 0$ for all integers u and v such that $u \geq 1$ and $v \geq 1$ are both true. Hence our long-distance factor (5.38) has no contributions at all from any configurations where a group of mono-bands, all on the same W_i, and not entangled with any other band, in the sense defined above, are sufficiently small in configuration space, that all their propagators have $|x-y| \leq T$. Now by power-counting, these are the only configurations which can produce linear divergences along the loops W_i, hence our long-distance factor (5.38) is completely free from such linear divergences.

Now these linear divergences, which we have just verified all cancel out of (5.38), are all associated with subdiagrams, of Feynman diagrams contributing to $f_0(W_1, \ldots, W_n, g^2)$, that have precisely two "path legs", and *no* gluon legs. Our long-distance factors (5.38) *do* still have residual divergences on the W_i's associated with subdiagrams, of Feynman diagrams contributing to $f_0(W_1, \ldots, W_n, g^2)$, that have precisely two "path legs", and *one* gluon leg. These subdiagrams look like "vertex corrections" to a vertex obtained by contracting the subdiagram to a point, so that the gluon leg ends directly on the path. We find by power-counting that these divergences are at worst logarithmic, and that they are *not* affected by the presence of corners in the path. (The logarithmic "corner divergences" discussed in [43] are associated with subdiagrams with *no* gluon legs, and, being entirely due to configurations where a group of mono-bands on a single W_i, not entangled with any other band, are such that all their propagators are "shorter" than T, they are totally absent from (5.38) by the proof just given.) These "vertex" divergences, due to subdiagrams with one gluon leg and two "path legs", simply give a fixed infinite renormalization of the gluon field $A_{\mu a}^{(x)}$ at the vertex, in the diagram obtained from the given diagram by contracting the divergent subdiagram to a point, where the gluon leg meets the path. However, in our BPHZ approach, there are no infinite renormalizations of the fields, and we therefore have to introduce counterterms for these divergent subdiagrams by hand. The counterterm for each such divergent subdiagram is defined once only: for an infinite straight path. This one counterterm cancels the divergence even when the divergent subdiagram is "going round a corner" - this follows directly from power-counting, since when we calculate the change in the counterterm that would be required near a corner, the "contraction point" of the subdiagram, (e.g. the mean position of the vertices on the path, or the position of one of the vertices on the path), is no longer fixed, hence we get one extra integral along the path, which makes the "corner discrepancy" finite. These extra counterterms are defined in our standard manner [54]. We have to choose a set of contraction weights for the subdiagram - the fact that the divergence is only logarithmic means that the counterterm is independent of the choice of the set of contraction weights. (In general, for a translation-invariant theory, changing the set of contraction weights changes the counterterm by a total divergence, but when the degree of divergence is zero, changing the set of contraction weights has no effect on the counterterm.) The propagators in these extra counterterms are cut off smoothly at long distances in exactly the same manner as the propagators in the usual coun-

terterms, with the onsets of the cutoffs occurring at $|x-y| = R$. We also have to add *finite* counterterms, which are necessary to ensure that, when we substitute the *perturbative* expansions of the long-distance factors of the windows into the right-hand sides of the group-variation equations, we obtain the correct renormalized perturbative expansions of the left-hand sides of the group-variation equations. These extra finite counterterms may be calculated in perturbation theory. They make their appearance in the solution of the group-variation equations via the *initial* valules of the f_0's, at small g^2, (or equivalently, at small L), which are also calculated in perturbation theory. They also have to be included in some non-island diagrams.

Now these counterterms, which we require for these subdiagrams with one gluon leg and two "path legs", do *not* arise from any counterterms in the action. They are special counterterms which, in our BPHZ approach, form an integral part of the definition of the long-distance factors of Wilson-loop vacuum expectation values and correlation functions. Nevertheless, we find that, if we change R to another finite value, R_2, also strictly greater than zero, the effect on these counterterms is exactly equivalent to leaving R *unaltered*, and multiplying by $A_{\mu a}^{(x)}$, in the path-ordered phase factor, by exactly the same finite factor $\frac{Z_1}{Z_3}$ as occurs in (5.27). (The finite change in the counterterm for a given subdiagram with two "path legs" and one gluon leg, contributes to the $\frac{Z_1}{Z_3}$ factor for the $A_{\mu a}^{(x)}$ at the vertex formed by contracting that subdiagram to a point.) Hence we can completely cancel the effects of the rescaling of $A_{\mu a}^{(x)}$ in (5.27) and in the path-ordered phase factor, by simply rescaling the integration variable $A_{\mu a}^{(x)}$ in the functional integral. This is the reason, in our approach, why there are no "extra terms" associated with field rescalings in the renormalization group equations (3.19) and (5.30) for our f_0's: the only "other terms" are those associated with the short-distance factors.

We note here one further crucial effect of dividing out the short-distance factors. We have to give a convergent limiting procedure for calculating the path integrals in the right-hand sides of the group-variation equations, (i.e. the sums over the 45-paths), in such a way that if, having restored the *short*-distance factors in the right-hand side windows as described above, we then substitute for each right-hand side window's *long*-distance factor (5.38), the *perturbative* expansion of that long-distance factor, we exactly recover, in the limit of our convergent procedure for calculating the path integrals, the renormalized perturbative expansion of the left-hand side of that group-variation equation. (This is, of course, the essential test of the validity of our limiting

procedure for calculating the path integrals.) The property of the long-distance factors (5.38) that makes this possible, is that the dependence of the long-distance factors on the details of the path is *soft*: the long-distance factors do not depend sensitively on the details of the path, even when that path is quite jagged. (All our paths are formed from finite numbers of straight segments.) This soft dependence of the long-distance factors on the details of the path is due to the $\frac{\mathrm{d}x_{\mu_i}(s_i)}{\mathrm{d}s_i}$ factors in the definition (1.1) of the path-ordered phase factor: these factors tend to average out the details of the path, provided the dependence, on $x(s_1), \ldots, x(s_n)$, of whatever expectation value $A^{(x(s_1))}_{\mu_1 a_1} \ldots A^{(x(s_n))}_{\mu_n a_n}$ is involved in, is not too singular.

5.6 Limiting procedure for calculating the path integrals

Our limiting procedure for calculating the path integrals consists, in brief, of the following. We first restore the short-distance factors, as sums of renormalized Feynman diagrams, with their propagators cut off smoothly at long distances, with the onsetof the long-distance cutoffs occurring at $|x - y| = T$, in the windows of the right-hand side group-variation equation diagrams, and also divide by the short-distance factors for the loops of the left-hand side f_0. We then express all the path sums for the *gluon* 45-paths in terms of $\frac{1}{D^2}$ by means of (1.16), (1.18), (1.20), and (1.30). (The path sums for the Fadeev-Popov 45-paths are given directly by $\frac{1}{D^2}$, as we see from the first term in (1.23).) We then make a sequence of approximations to $\frac{1}{D^2}$, as follows. At *every* stage in our sequence of approximations, we now go to the limit of zero segment lenght, or in other words, zero width of each individual Gaussian. *But*, we do not "tie" the *long*-distance factor to the path all the way along its length: we only tie the *short*-distance factor precisely to the path all along its length. Our approach is the same as before, (equations (1.24) - (1.26)), except that the parameter σ that characterizes each approximation now represents no the "maximum tolerable" Gaussian width of an individual segment, but rather the "maximum tolerable" *total* of all the infinitesimal Gaussian widths between successive points where we "tie" the long-distance factor to the path. (By "Gaussian width" we mean the actual coefficient of $\bar{D}^2$ in an exponent, *not* the square root of that coefficient.) What this means in practice is that, for the σ-approximation to $\frac{1}{D^2}$, we break up the s-integral in equation (1.24) as in equation

(1.25) as before. then for $n \geq 1$, and for $n\sigma \leq s \leq (n+1)\sigma$, we break up $\left(e^{s\bar{D}^2}\right)_{Ax,By}$ into a product of $(n+1)$ factors, exactly as before in equation (1.26). Now equation (1.26) is exact. What we now do is, for each of the factors $\left(e^{\frac{s\bar{D}^2}{n+1}}\right)_{C_i z_i, C_{i+1} z_{i+1}}$ in (1.26), (where $C_0 z_0 \equiv Ax$ and $C_{n+1} z_{n+1} \equiv By$), to represent this factor again by a path integral, but, bearing in mind that the Gaussian width $\frac{s}{n+1}$ of this factor is $\leq \sigma$, we approximate the *long*-distance factor's path, for *every* path in this path integral, by the straight line from z_i to z_{i+1}. The *short*-distance factor follows the true path exactly. The result is that, for insertion into (1.26), we approximate $\left(e^{\frac{s\bar{D}^2}{n+1}}\right)_{C_i z_i, C_{i+1}, z_{i+1}}$ by the straight line path from z_i to z_{i+1} for the *long*-distance factor, times:

$$\sum_{m=0}^{\infty} \int_0^{\infty} \dots \int_0^{\infty} \mathrm{d}t_1 \dots \mathrm{d}t_m \delta\left(\frac{s}{n+1} - (t_1 + \dots + t_m)\right) \quad \times$$

$$\times \left(e^{t_m \partial^2} (\partial A + A\partial + AA) e^{t_{m-1}\partial^2} (\partial A + A\partial + AA) \dots (\partial A + A\partial + AA) e^{t_1 \partial^2}\right)_{C_i z_i, C_{i+1} z_{i+1}} \tag{5.54}$$

where $A \equiv A_{\mu a}^{(x)}(t_a)_{BC}$ here is an effective field in terms of which we represent the short-distance factor. Thus at *every* stage of our sequence of approximations to the path integral, we treat the *short*-distance factor exactly: it is only the *long*-distance factor whose path is approximated by $(n+1)$ straight segments in (1.26). Thus at every stage in our sequence of approximations, we retain the correct short-distance behaviour of the underlying renormalized Feynman diagrams. Our procedure converges due to the soft dependence of the long-distance factors on the details of the paths. We do have to make special allowance, however, for cases where two *different* loops in the border of one of our non-simply connected windows approach one another closely, for in such a case, as we noted before, the short-distance divergence is in the long-distance factor, *not* the short-distance factor. However, due to the fact that we never have more than one island, and that if we do have an island, there are no 45-paths that do *not* form part of that island, we *never*, in the right-hand sides of the group-variation equations, have to do a path-integral for more than one of the connected borders of a non-simply connected window. Thus the only configurations that need special attention in this regard are those where a border of an island intersects, or comes close to, one of the loops in the left-hand side f_0. For details and renormalization, we refer to our next paper.

Chapter 6

Convergence Of The Sums In The Right-Hand Sides, Renormalization In Position Space, And The Signs Of The Island Diagrams

6.1 Renormalization group equations for the ansatz parameters

We now return to the discussion, begun before, of the dependence of μ, where μ^2 is the coefficient of the area in the Wilson area law in our ansatz, on our input parameters g^2 and R. Our long-distance factors (5.38) satisfy, as we stated before, a renormalization group equation of the form (5.30), where the "other terms" are associated exclusively with the short-distance factors which we divide $f_0(W_1, \ldots, W_n, g^2)$ by in (5.38). Now these "extra terms" involve the partial derivative $\frac{\partial}{\partial T}$, where T is the onset value $|x-y|$ of the long-distance cutoffs imposed on the propagators in the $\tilde{f}_0$'s in the denominator of (5.38). Our ansatz does not contain T at all, because it only refers to the essential features of the long-distance behaviour of our f_0's. The T-dependence of our long-distance factors does not interfere at all with these essential features: as we know, the T-dependence of our long-distance factors (5.38) is constrained by the fact that if we multiply these long-distance factors by the (finite) ratio of the short-distance factors for T divided by the short-distance factors for another value, say T_2, the dependence

of this product on T must completely cancel out, to be replaced by the equivalent dependence on T_2. It follows that the factors in our ansatz obey (5.30) exactly with *no* "other terms" at all. In particular, for a single Wilson loop, we have:

$$\left(-R\frac{\partial}{\partial R}+\beta(g)\frac{\partial}{\partial g}\right)e^{-\mu^2 A}=0 \tag{6.1}$$

Now A, the area of the minimal-area spanning surface of our loop, is of course completely independent of R and g, hence (6.1) implies:

$$-R\frac{\partial\mu}{\partial R}+\beta(g)\frac{\partial\mu}{\partial g}=0 \tag{6.2}$$

And on dimensional grounds, the dependence of μ on R is simply by an overall factor of $\frac{1}{R}$, (so that μR is a function only of g^2), hence (6.2) implies:

$$\mu R+\beta(g)\frac{\mathrm{d}(\mu R)}{\mathrm{d}g}=0 \tag{6.3}$$

Hence, comparing with equation (5.30), we see that the functional dependence of μR on g^2 is identical in form to a "curve of constant f" in the (R,g) plane, which in other words, means that the functional relationship between g^2 and μR is identical to a possible behaviour of the "running coupling" $\bar{g}^2(R)$. From equation (5.36) and diagram (5.37) we therefore see that for small g^2, μR decreases very rapidly as g^2 decreases towards zero, hence, as stated, $\frac{1}{\mu R}$, which is the typical island size measured in units of R, *increases* very rapidly as the input value of g^2 decreases towards zero.

6.2 Convergence of the sums in the right-hand sides of the Group-Variation Equations

We note here that, although, as we saw above, the experimental value $\alpha_s = 0.3$ for charmonium implies that the critical value of $\frac{g^2}{4\pi}$, as given by equation (5.10), must be greater than 0.35, we cannot get a good indication of the actual critical value from experimental results, since the experimental value of μ, which is approximately where the running coupling reaches the critical value, is 0.41 GeV, while quoted values of Λ_s range from 0.1 GeV to 0.5 GeV [55]. [1] Furthermore, the critical value will be renormalization-scheme dependent. (That is not a serious problem, however, because

[1] More recent experimental data show Λ_s stabilizing around 0.2 GeV [5].

it is easy to specify a simple and natural renormalization scheme. For example, we can require that our additional Ward-identity-restoring finite counterterms contain no $A^{(x)}_{\mu a}\partial_\mu\partial_\nu A^{(x)}_{\nu a}$ terms, which is the choice made in (5.12).) The actual calculation of the critical value of g^2 from (5.10) requires the full renormalization of the group-variation equations, which we treat in our next paper, and we therefore turn now to the following consideration.

The order g^6 term in $2g\beta(g)$ is known [56], [57], [58], [59], and like the order g^4 term, it is renormalization-scheme-independent [60]. From the quoted results, we find that our $\beta(g)$ is given by:

$$2g\beta(g) = -\frac{11g^4}{12\pi^2} - \frac{17g^6}{48\pi^4} + \text{ terms of order } g^8 \tag{6.4}$$

Now of course, it is possible to find a renormalization scheme where, by adding suitable *extra* finite counterterms at each order in the loop expansion of the effective action Γ, every term in $\beta(g)$ beyond the two displayed above, vanishes. Such a renormalization scheme would be, from the point of view of the group-variation equations, extremely unnatural and undesirable. it seems much more likely that, in a natural renormalization scheme, such as that suggested above, where we fix the Ward-identity-restoring finite counterterms by requiring them to include no term $A^{(x)}_{\mu a}\partial_\mu\partial_\nu A^{(x)}_{\nu a}$, allow no other finite counterterms, and at each order in the loop expansion of Γ, choose the BPHZ counterterms to have the canonical form given in our previous paper [26], as generated by the seed action plus all the lower-order finite counterterms, the expansion coefficients in $2g\beta(g)$ will have roughly the same behaviour as the expansion coefficients in the expansions of other physical quantities in powers of g^2. Let us assume a weak form of this hypothesis, namely that, in a natural renormalization scheme, such as that suggested above, the expansion coefficients in $2g\beta(g)$ do *not* behave *better*, (with regard to convergence), than the expansion coefficients in the expansions of other physical quantities, in the *explicit* powers of g^2 that multiply the diagrams in the right-hand sides of the group-variation equations, (namely, for each diagram, a power of g^2 equal to its number of 45-paths minus its number of action vertices), and also the explicit powers of g^2 in the short-distance factors, when we restore them in the windows of the right-hand side group-variation equation diagrams. Now of course, these expansion coefficients depend on the ansatz for the long-distance factors which we substitute into the right-hand side of the group-variation equations, so to make the hypothesis precise, we assume it applies with the ansatz for the long-distance factors we discussed above,

when that ansatz is modified to include the correct dependence on the onset length T of the smooth long-distance cutoffs impose on the propagators in the short-distance factors we divide by in (5.38). (Note added: of course, we also assume it is true when the actual solution of the group-variation equations is substituted into the right-hand sides of the group-variation equations.)

We next note that, for planar diagram field theories, such as large-N_c QCD, in four Euclidean dimensions, 't Hooft [6] [7] has shown that, firstly, if there are no divergent subdiagrams, then the perturbation-theory expansion coefficients grow at worst geometrically, and there exists a circle of convergence in the g^2-plane, of radius strictly greater than zero, inside which the expansions, in powers of g^2, of *all* physical quantities converge, and, secondly, if the theory is asymptotically free when the divergent subdiagrams are included, the input value of g^2 is sufficiently small, that there exists a mass to terminate the growth of the running coupling at large distances, then there exists an absolutely convergent procedure, based on skeleton expansions, difference equations in momentum space, and the renormalization group, for calculating *all* physical quantities.

Now the lower bounds on the radii of the circles of convergence obtained by 't Hooft are several orders of magnitude smaller than the experimental lower bound, 0.35, on the critical value of $\frac{g^2}{4\pi}$, but let us, nevertheless, suppose that the general behaviour of the expansion coefficients, in the expansions of physical quantities in the "explicit" powers of g^2, (as defined above), is not worse than geometric. Then by our hypothesis, the behaviour of the expansion coefficients in $2g\beta(g)$ is *at best* geometric, and, if the expansions of other physical quantities in the "explicit" powers of g^2 have a finite radius of convergence, then the expansion of $2g\beta(g)$ in powers of g^2 also has a finite radius of convergence, which is not greater than the radius of convergence for other physical quantities. Now from (6.4) we see that the first two coefficients in the expansion of $2g\beta(g)$ have the *same* sign. Let us suppose that this trend continues, and that in a natural renormalization scheme, *all* the expansion coefficients in $2g\beta(g)$ have the *same* sign. Then if the expansion of $2g\beta(g)$ in powers of g^2 has a finite radius of convergence, $2g\beta(g)$ will tend to $-\infty$ as g^2 approaches the radius of convergence from below along the positive real axis. This means that the "curves of constant f" in diagram (5.37) will curl upwards to a vertical slope at the finite value of g^2 given by the radius of convergence. That does not matter, because from equation (5.10) we see

that, *assuming* that $\sum_i Y_i$ is negative, (we return to this point shortly), if $\frac{\beta(g)}{g}$ goes to $-\infty$ for a finite value of g^2, strictly greater than zero, then the critical value of g^2, as determined by (5.10), will be strictly smaller than the value of g^2 where $\frac{\beta(g)}{g}$ goes to $-\infty$. This is so because the right-hand side of (5.10) is on dimensional grounds equal to μ^2, times a function of g^2, hence if we divide both sides of (5.10) by μ^2, we have in the right-hand side a function of g^2 which sweeps out all real values from 0 to $+\infty$ as g^2 increases from 0 to the radius of convergence. (The leading terms in $\sum_i Y_i$ come from the one-loop island diagrams and are independent of g^2. [2] There is certainly no reason to expect $\sum_i Y_i$ to vanish as $2g\beta(g)$ goes to $-\infty$.)

Now by assumption, the expansion coefficients in $2g\beta(g)$ do *not* behave *better* than the expansion coefficients in the expansions of other physical quantities in the "explicit" powers of g^2, as defined above, hence it follows immediately that if, in a natural renormalization scheme, the trend in (6.4) continues, and all the expansion coefficients in $2g\beta(g)$ have the same sign, then at the critical value of g^2, as determined by (5.10), the expansions of physical quantities, (such as the right-hand sides of the group-variation equations, for a reasonable ansatz), in the "explicit" powers of g^2, will *converge*, and the convergence rate will be at least geometrical. In other words, if all the expansion coefficients in $2g\beta(g)$ have the same sign, then the singularity in $\frac{\beta(g)}{g}$ will shield all other physical quantities from having divergent expansions, by ensuring that the critical value of g^2 is strictly smaller than the radius of convergence of the expansion of any physical quantity in the "explicit" powers of g^2.

As a check we note that, if we assume that the two terms given in (6.4) are the first two terms of a geometric series, then the radius of convergence is given by $\frac{g^2}{4\pi} = \frac{11\pi}{17} = 2.03$, which is safely larger than the experimental lower bound of 0.35 on the critical value of $\frac{g^2}{4\pi}$.

We note here that, with a simple derivative in the left-hand sides, and integrals in the right-hand sides, the group-variation equations have the ideal form for solving by repeated substitution of the left-hand sides, after integrating with respect to L or g^2, from initial conditions given for small L or small g^2 by perturbation theory, into the right-hand sides. If a metric can be found in the function space of our long-distance factors (5.38), such that if when we substitute two different ansätze into the right-hand sides of the group-variation equations, the distance in function space between the two

[2] Note added: that is, they have no explicit dependence on g^2.

"outputs" is strictly less than the distance in function space between the two "inputs", then this procedure will converge to a unique solution of the group-variation equations.

The considerations just given indicate that it is likely that, if a reasonable ansatz is substituted into the right-hand sides of the group-variation equations, and the input value of g^2 is not larger than the critical value, then the sums over the diagrams in the right-hand sides of the group-variation equations will converge.

We may, in practice, want to consider iterating the group-variation equations with truncated versions of their right-hand sides, and then repeating the process with less truncated versions of their right-hand sides, and so on, and it is likely that such a procedure would also converge.

6.2.1 Application of the four-loop β-function

From the general form of the four-loop β-function in $\overline{\text{MS}}$, given in reference [9], we find that our $\frac{\beta(g)}{g}$ is given by:

$$\frac{\beta(g)}{g} = -\left(\frac{11}{3}\frac{g^2}{8\pi^2} + \frac{34}{3}\left(\frac{g^2}{8\pi^2}\right)^2 + \frac{2857}{54}\left(\frac{g^2}{8\pi^2}\right)^3 + 315.49\left(\frac{g^2}{8\pi^2}\right)^4 + \text{order}\left(g^{10}\right)\right) =$$

$$= -\left(0.5836\frac{g^2}{4\pi} + 0.2871\left(\frac{g^2}{4\pi}\right)^2 + 0.2133\left(\frac{g^2}{4\pi}\right)^3 + 0.2024\left(\frac{g^2}{4\pi}\right)^4 + \text{order}\left(g^{10}\right)\right) \tag{6.5}$$

(We recall that, as explained in Subsection 5.4.5, our $\frac{g^2}{4\pi}$ is equal to $\frac{3}{2}$ times the value α_s would have in the absence of quarks.) The coefficients are decreasing, but that is an artifact of our choice of expansion parameter. The ratios of successive pairs of coefficients in the expansion are increasing, but at a decreasing rate. If we assume that the difference, between two successive pairs of ratios, changes by a fixed factor, for each successive pair of ratios, then the ratio, of two successive coefficients, tends asymptotically to 1.89, which means that the series will diverge when $\frac{g^2}{4\pi} = 0.53$. And if we assume no more than that the ratios of successive pairs of coefficients will go on increasing, then we can conclude that the series will diverge for $\frac{g^2}{4\pi} \leq 1.05$. The four-loop term is essential to reach these conclusions.

6.3 BPHZ renormalization in position space

We think of our BPHZ counterterms [66], [67], [68], [69], [26], as enormously deep reservoirs, whose actual depth, (at small distances), is unknown, and of no interest. We only see the surface of the reservoir, (at large distances), whose height corresponds to $\ln(R)$. And the fact that, way down in its depths, our reservoir may have irregularities and caverns that do not correspond to the simple BRST-invariance properties of its surface, is of no concern to us at all. Now in correspondence with Theorem 2 of our previous paper [61], we do indeed assume that the depths of our reservoirs, though enormous, are finite, and that the amount of irregularity is limited, (the propagators in all our counterterms and "direct terms" are *smoothly* regularized at extremely short distances). We also assume that the deep structures of our "positive" and "negative" reservoirs, (the counterterms and the direct terms), although unknown, exactly match. We then find that the calculated values of observable quantities, above or close to the surfaces of the reservoirs, have finite limits as the depths of the reservoirs tend to ∞. In our BPHZ approach, we *never* have to refer to the actual depths of the reservoirs, because we always add the contributions of a positive reservoir and a negative reservoir with the same deep structure before calculating a volume. The reason we have to assume that the amount of irregularity at great depths is limited, (propagators are *smoothly* regularized at extremely small distances), is that when we work in position space, certain reservoirs, namely those associated with one-line *reducible* subdiagrams with two, three, or four legs, can appear superficially, (i.e. by power-counting), to have infinite volumes, when in fact their volumes are finite. The calculation of the volumes of these reservoirs in position space is only conditionally convergent, but there is only one natural way to approach the limit: namely from propagators *smoothly* regularized at exremely small distances. *Any* smooth short-distance regularization satisfying the general conditions of Theorem 2 of our previous paper [61] will give the same result for the volume. The reason for this is that, as noted on page 236 of that paper, the actual uses made of the regularization are extremely limited: they amount to allowing us to sweep derivatives past the "key" propagators of one-line-reducible subdiagrams by integrations by parts and translation invariance, without picking up any "short distance surface terms" due to having to cut a small sphere centred at one end of such a propagator, out of the integration domain of the other end of such a propagator. Once these manipulations are done, absolute convergence is attained, and the smooth

regularization is no longer necessary. Examples of smooth regularizations with the required properties are constructed on pages 235 and 236 of that reference: we never need to use their detailed forms, it is sufficient that they exist.

We call these enormously deep reservoirs, whether they occur in the counterterms or in the "direct" parts of our calcultions, and whose depths are unknown but must match between different parts of our calculations, "counters", (analogous to "differentials").

We in fact did *not* assume translation invariance in the above reference: only a much weaker assumption, called "translation smoothness", was made. Defined on pages 224 and 225 of that paper, and included in a general form in the conditions for Theorem 2, translation smoothness essentially requires the power-counting behaviour of a propagator as y tends to x to be independent of x, or at least not worse than the standard translation-invariant power-counting behaviour, while allowing the *coefficient* of the singularity as y tends to x, to depend smoothly on x. (Examples of translation smooth propagators are given by ordinary propagators, multiplied by completely smooth functions of x and y.) Thus the question of whether translation invariance is necessary for BPHZ finiteness has been answered: translation invariance is *not* necessary, translation smoothness is sufficient. We note, however, that while for translation invariant theories, changing the choice of the contraction weights [54] used to define a counterterm, (i.e. the weights assigned to the vertices of the "direct term", in the linear combination of the positions of those vertices, that defines the position of the contraction point of the counterterm), results in changing the counterterm only by a total derivative, and in fact leaves the counterterm unaltered if the degree of divergence is zero, we do not find a corresponding result in non-translation-invariant theories. Thus the choice of the contraction weights becomes a dynamical issue in non-translation-invariant situations, for example in the presence of background gravitational fields.

When one forms the *position-space* integrand for a particular Feynman diagram and a particular BPHZ forest for that Feynman diagram, one has to detach the inner ends of the legs, of each subdiagram that is a member of that forest, from the Feynman diagram vertices they are initially attached to, and move them to the position of the contraction point of that subdiagram. The contraction point of a subdiagram is a linear combination of the positions of the vertices of the subdiagram, for example their mean position. Other choices are more convenient in practice, for example one can number all the vertices of the Feynman diagram, and define the contraction point of each subdiagram to be the highest-numbered vertex of that subdiagram. The choice

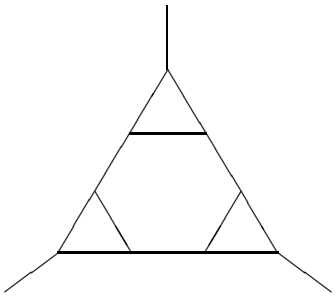
of the contraction points of members of the forest must satisfy the following constraint [54]: the contraction point of each member of the forest must be expressible as a linear combination of the contraction points of the largest members of the forest that are strict subsets of that member of the forest, plus the positions of any vertices of that subdiagram that are not vertices of any strictly smaller member of the forest. Both the "mean position" choice and the "highest-numbered vertex" choice that we just mentioned, satisfy this criterion. (The reason we define "greenwoods" in [24] and "woods" in [26], which are in one-to-one correspondence with the usual forests, is to avoid having to make frequent separate references to "the vertices of that subdiagram that are not vertices of any strictly smaller member of that forest".)

It is very helpful in position-space BPHZ to have a separate index or label for every propagator end. We can then represent each vertex of the Feynman diagram by the set of all the propagator ends at that vertex. The set V of all the vertices of the Feynman diagram then becomes a partition of the set of all the propagator ends. In [26] we represent the set V of all the vertices of a Feynman diagram by a partition of a set that includes all the propagator ends, but may have other members as well. The advantage of this is that not only all subdiagrams, but also all diagrams obtained from the given diagram by *contracting* some of its subdiagrams, can be represented in exactly the same way. In fact, diagrams obtained from the given diagram by contracting some of its subdiagrams, can be represented by partitions of exactly the same set as the given diagram.

The operation of moving the inner ends of the legs of the subdiagrams that are members of the forest, to the contraction points of those subdiagrams, is performed in a sequence in which smaller members of the forest precede any larger members of the forest that contain them. The final result is that each propagator end is moved to the contraction point of the largest member of the forest that has that propagator end as a member, but does not have the other end of that propagator as a member. If a propagator end is not a membmer of any member of the forest, it is left in its original position. (Again, working with woods enables us to avoid having to state this last case separately.)

These movements of propagator ends, which are an essential part of BPHZ in position space, cause the following practical problem: to ensure that our BPHZ-renormalized feynman diagrams are convergent in position space, we want to use the Cluster Convergence Theorem [24], which involves cutting up a position space integrand

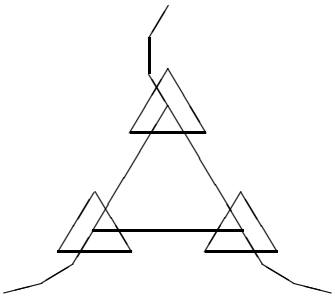
into a finite number of sectors, characterized by the way the vertices are clustered in position space. If V is a finite set, x is a map from V into position space, and σ is a real number such that $0 < \sigma < 1$, then a σ-cluster of x is a nonempty subset S of V such that for all $i \in S$, $j \in S$, $k \in S$, and $l \in V$ such that l is *not* a member of S, $|x_i - x_j| < \sigma\,|x_k - x_l|$. The set of all the σ-clusters of x is a forest of V. For reasons of convenience, we also explicitly specify that *all* one-member subsets of V are σ-clusters of x. The set of all the σ-clusters of x is then a forest of V that includes all the one-member subsets of V. This is called a greenwood of V in [24]. The Cluster Convergence Theorem then guarantees absolute convergence if for every greenwood G of V, the absolute value of the integrand can be bounded, throughout the secto of configuration space where the set of all the σ-clusters of x is equal to G, by a constant times a product of powers of the distances between the vertices, that satisfies a standard power-counting condition for every member of G that has two or more members. The problem caused by the movements of propagator ends, as described above, for the different BPHZ forest, is that for a given position-space configuration of the vertices of the original Feynman diagram, the clusters defined by the positions of the contraction points of the members of a forest, may be completely different for different forests, and completely different from the clusters defined by the positions of the vertices in the original Feynman diagram. For example, if we have a triangle subdiagram that has a separate vertex insertion in each of its three corners:



(6.6)

then the mean positions of the vertices of each of the three "corner" triangles can be close to one another in position space even if *none* of the nine vertices of the original subdiagram are close to one another in position space. This means that, if we are using the "mean position" choice of contraction points, and we consider a BPHZ forest that includes the three corner triangles as members, we will find, after moving, for each corner triangle, the "inner ends" of the legs of that triangle to the contraction point of

that triangle, as represented symbolically here:



(6.7)

that we now have a "cluster" configuration in position space for this subdiagram, due to the assumed closeness, in position space, of the mean positions of these three corner triangles to one another, even though *none* of the nine vertices of the original triangle are close to one another in position space. This means that to decide, for each position-space configuration of the vertices of the original Feynman diagram, which BPHZ forests' contributions need to be combined together to obtain integrands which satisfy the conditions for applying the Cluster Convergence Theorem, it is *not* good enough just to look at the position-space clusters in the originial Feynman diagram: we have to look separately at the position-space clusters for each individual BPHZ forest, since the precise set of contraction points is different for each BPHZ forest. But on the other hand, for each possible greenwood of position-space clusters of the vertices, we have to combing the BPHZ forest into groups, called "good sets of woods", such that for each group, the sum of the integrands of the members of that group, satisfies a bound that lets us apply the Cluster Convergence Theorem. The appropriate groupings are defined by pairs (P, Q) of forests or woods, such that $P \subseteq Q$, and the members of the grouping defined by the pair (P, Q) are all the forests, or woods, F, such that $P \subseteq F \subseteq Q$. But, given a configuration x of the vertices of the original Feynman diagram, (i.e. a map from V into position space), and a BPHZ forest F, how should we decide to which pair (P, Q) the forest F should be assigned? The improvement of short-distance behaviour obtained by adding together the integrands of all the BPHZ forests in the group defined by the pair (P, Q), occurs for the members of Q that are not members of P. This is because the group of BPHZ forests defined by the pair (P, Q) includes precisely all the BPHZ forests that include every member of P, and, independently for each member of Q that is *not* a member of P, may or may not include that member of Q. We get an extra minus sign for each extra member of Q included, and the resulting cancellations have the effect of shifting enough denominator posers of intervals out from the members of Q that are not members of P, onto the legs of those

members of Q, to enable the members of Q that are not members of P, to satisfy the conditions for applying the Cluster Convergence Theorem. Thus to obtain manifest convergence we need to ensure that every position-space cluster that is power-counting divergent in the original Feynman diagram, becomes a member of Q that is not a member of P, and also that no member of P is small in position space. Here "cluster" and "small in position space" refer, for each subdiagram that is a member of F, to what we have after contracting all members of F that are strict subsets of that member of F. Let us define, for any BPHZ forest F, subdiagram A, and configuration x of the vertices of the original Feynman diagram, $\mathbf{L}(F, A, x)$ to be the largest distance between any two vertices of A, *after* all members of F that are strict subsets of A, have been contracted. (So in other words, $\mathbf{L}(F, A, x)$ is the "diameter" of A, *after* all members of F that are strict subsets of A, have been contracted.) Now the example above shows that $\mathbf{L}(F, A, x)$ can be very small even when *none* of the distances between pairs of vertices of A are small in the original Feynman diagram. There is another possibility that can also affect the identification of the position-space clusters. A σ-cluster is a subset of the vertices whose "diameter" , in a particular position-space configuration, is less than σ times the smalles distance between any member of the σ-cluster and any vertex that is *not* a member of the σ-cluster. It is possible for the contraction of a subdiagram to put a vertex, namely the contraction point of that subdiagram, near a set of vertices that previously had no other vertex near them. Hence to identify the forests P and Q, which define the "good grouping of forests" to which we should assign the BPHZ forest F, for a given configuration x of the vertices of the original Feynman diagram, we have to take account, for each subdiagram A, not only of the contractions of all members of F that are strict subsets of A, (and hence use $\mathbf{L}(F, A, x)$ rather than the diameter of A in the original Feynman diagram), but also to take account of the contractions of members of F that are *disjoint* from A. To do this we note that, as mentioned above, it is very helpful to have a separate index or label for every propagator end. We can then represent each propagator by the two-member set whose members are the two ends of that propagator. Now as we noted before, after doing all the contractions for a particular BPHZ forest, each propagator end has moved to the contraction point of the largest member of the forest that has that propagator end as a member, but does not have the other end of that propagator as a member. We can express this by defining, first, the set H to be the set of all the propagators, and, secondly, for each BPHZ forest F and propagator end i, $\mathcal{Z}(F, H, i)$ to be the largest

member of F that has i as a member but does *not* have the other end of that propagator as a member. We also define, for every subdiagram A, x_A to be the contraction point of A.

Thus the simplest reasonable generalization of the σ-cluster criterion to the case where we have to take into account the contractions associated with the members of the BPHZ forest F, would be to require that for every propagator end i that is a member of A, and every propagator end j that is *not* a member of A, we have the following inequality:

$$\mathbf{L}(F, A, x) < \sigma \left| x_{\mathcal{Z}(F,H,i)} - x_{\mathcal{Z}(F,H,j)} \right| \tag{6.8}$$

However this is over-stringent: there is no point in applying such a restriction when the propagator end j is in some part of the Feynman diagram that has no direct connection at all to the subdiagram A. Thus we only require (6.8) ot be satisfied when $\{i, j\}$ is a propagator such that i is a member of A and j is *not* a member of A.

Now we are here considering the case of Theorem 1 of [26], where we allow counterterms for *all* power-counting divergent connecte subdiagrams, both one-line reducible and one-line irreducible. (The additional considerations necessary when only one-line *irreducible* counterterms are allowed, are given in Theorem 2 of [26].)

We therefore try the following identification of the pair P and Q of forests that defines the "good grouping of forests" to which we should assign F, for the given configuration x of the vertices of the original Feynman diagram:

$$\left.\begin{array}{l} P \text{ is the set of all the members } A \text{ of } F \text{ such that there} \\ \text{exists a propagator } \{i, j\} \text{ such that } i \text{ is a member of } A, \\ j \text{ is } \textit{not} \text{ a member of } A, \text{ and (6.8) is } \textit{not} \text{ true.} \end{array}\right\} \tag{6.9}$$

$$\left.\begin{array}{l} Q \text{ is the union of } F, \text{ and the set of all the power-counting} \\ \text{divergent connected subdiagrams } A \text{ that overlap no member} \\ F, \text{ such that (6.8) is true for } \textit{all} \text{ propagators } \{i, j\} \text{ such} \\ \text{that } i \text{ is a member of } A \text{ and } j \text{ is } \textit{not} \text{ a member of } A. \end{array}\right\} \tag{6.10}$$

Now (6.9) and (6.10) certainly give a pair (P, Q) such that $P \subseteq F \subseteq Q$ is true, and we see that, roughly speaking, the members of P are the members of F that are *not* in "position-space cluster configurations" after the contractions for the members of F have been done, while the members of Q that are *not* members of P, (and will hence get their short-distance behaviour improved when we combine the contributions of all the BPHZ forests in the "good set of forests defined by the pair (P, Q)), are,

roughly speaking, all the power-counting divergent connected subdiagrams that do *not* overlap any member of F, and which *are* in "position-space cluster configurations" after the contractions for the members of F have been done. But we now have to ask the following questions:
1) Is Q a forest? In other words, do we have a guarantee that none of the members of Q overlap one another?
2) Suppose Q *is* a forest. Do we have a guarantee that for *every* member G of the "good set of forests" defined by the pair (P, Q), or in other words, for *every* forest G such that $P \subseteq G \subseteq Q$, if we calculate a forest P' and a set Q' from (6.9) and (6.10) with F replaced by G, we will find that $P' = P$ and $Q' = Q$?

Now our purpose is to make the convergence, in position space, of our BPHZ-renormalized Feynman diagram manifest in the following way. We cut up the set of all ordered pairs (F, x) of a BPHZ forest F, and a configuration x of the vertices of the original Feynman diagram, into a finite number of sectors, each characterized by a pair (P, Q) of BPHZ forests such that $P \subseteq Q$. The sector associated with the pair (P, Q) of BPHZ forests, is to be equal to the Cartesian product of the set of all BPHZ forests F such that $P \subseteq F \subseteq Q$, and an appropriate sector of the configuration space of the vertices of the original Feynman diagram. Now when we add the integrands for all the BPHZ forests F such that $P \subseteq F \subseteq Q$, we get an improvement in the short-distance behaviour of all the subdiagrams that are members of Q but are *not* members of P. Thus the appropriate sector of the configuration space of the vertices of the pair (P, Q), is a sector where all the members of P, after performing the contractions for the members of P, are *not* "clusters", while the members of Q that are *not* members of P, after performing the contractions for the members of P, or maybe for the members of Q, *are* "clusters". There is an inherent ambiguity with regard to whether we should contract *all* the members of Q before identifying the clusters, or whether we should just contract the members of P before identifying the clusters. In other words, should we replace F in (6.8) by P or by Q? Now if we just want to identify the clusters "roughly", then this ambiguity doesn't matter, because since the members of Q that are *not* members of P are required to be small in position space whichever criterion we use, contracting them does not move any propagator end by a large distance.

But for our demonstration of manifest convergence to be valid, our sectors, associated with the pairs (P, Q) of BPHZ forests, must *exactly* tesselate the Cartesian product of the set of all the BPHZ forests and the set of all configurations of the ver-

tices of the original Feynman diagram. There must be no gaps or overlaps at all. (In the present case, where we allow counterterms for one-line-reducible subdiagrams as well as for one-line-irreducible subdiagrams, we obtain absolute convergence, and no integrations by parts are done, so provided the tesselation is exact, there is no problem with cutting up the integration domain into a finite number of sectors.)

The tesselation problem is solved by making the following requirement: we must give a precise rule by which, given any BPHZ forest F and any configuration x of the vertices of the original Feynman diagram, we can calculate uniquely the pair (P, Q) of BPHZ forests to whose sector the pair (F, x) belongs. Our first attempt at guessing such a rule is given above in (6.9) and (6.10).

Now of course the set Q we calculate by such a rule has got to be a BPHZ forest, hence for any acceptable such rule, the answer to question 1) above has got to be, "Yes".

Furthermore, we require that the sector associated with the pair (P, Q) of BPHZ forests, is to be the *Cartesian product* of the set of all forests F such that $P \subseteq F \subseteq Q$, and a certain domain of the configuration space of the vertices of the original Feynman diagram.

In other words, for each pair (P, Q) of BPHZ forests such that $P \subseteq Q$, and for each point x of the configuration space of the vertices of the original Feynman diagram, we require that either the pair (F, x) belongs to the sector defined by (P, Q) for *all* the forests F such that $P \subseteq F \subseteq Q$, or else the pair (F, x) belongs to the sector defined by (P, Q) for *none* of the forests F such that $P \subseteq F \subseteq Q$. It immediately follows from this, that for any acceptable rule for calculating P and Q, given F and x, the answer to question 2) above has also got to be, "Yes".

Now in fact, when Q is given by (6.10), the anwer to question 1) above is, "Yes". However, when P and Q are given by (6.9) and (6.10), the answer to question 2) above is, "No". Indeed, one can construct examples where, starting from the empty forest as F, one has two non-intersecting subdiagrams, both of which are members of Q by (6.10). However contracting one of the two subdiagrams moves the outer end of a leg of the second subdiagram towards the second subdiagram, and prevents the second subdiagram from being a member of Q according to (6.10), if we start from the forest whose only member is the first subdiagram, as F.

Instead of attempting to guess an acceptable improved version of (6.9) and (6.10), we now give a construction for which a "Yes" answer to question 2) above is automatic.

For each configuration x of the vertices of the original Feynman diagram, we define a set Ω, whose members are *all* the pairs (P, Q) of BPHZ forests such that $P \subseteq Q$ and, for every member A of Q that is not a member of P, there *exists* a forest F such that $P \subseteq F \subseteq Q$ is true, and (6.8) is true for all propagators $\{i, j\}$ such that i is a member of A and j is not a member of A. We note that the pair (F, F) is a member of Ω for *all* BPHZ forests F.

Now the members of Ω are, in a sense, all the pairs (P, Q) of BPHZ forests which might reasonably try to "claim" x as part of the configuration space domain in which they define a good set of forests. Two members (P, Q), and (P', Q') of Ω are in direct competition for x if $P \cup P' \subseteq Q \cap Q'$ is true, for in that case there exist BPHZ forests F, for example $P \cup P'$, such that both $P \subseteq F \subseteq Q$ and $P' \subseteq F \subseteq Q'$ are true. This competition is resolved in the following way. It turns out that if the fixed real number σ, which enters the definition of Ω by (6.8), satisfies $0 < \sigma \leq \frac{1}{8}$, then for any members (P, Q), and (P', Q') of Ω such that $P \cup P' \subseteq Q \cap Q'$ is true, the set $Q \cup Q'$ is a forest, (i.e. no member of Q overlaps any member of Q'), and the pair $(P \cap P', Q \cup Q')$ is also a member of Ω. This is proved in Lemmas 6 and 7 in [26]. It immediately follows from this result, as shown in Lemma 8 of [26], that if X is any nonempty subset of Ω, such that for any partition of X into two non-empty parts Y and Z, there exists a member (P, Q) of Y and a member (P', Q') of Z such that $P \cup P' \subseteq Q \cap Q'$ is true, then the union of the Q's of *all* the members of X is a forest, and the set $(\tilde{P}, \tilde{Q})$, where $\tilde{P}$ is the intersection of the P's of *all* the members of X, and $\tilde{Q}$ is the union of the Q's of *all* the members of X, is a member of Ω. Now the *maximal* such subsets X of Ω do not intersect one another, and the set of all of them forms a partition of Ω. Furthermore, as we noted above, for *every* BPHZ forest F, the pair (F, F) is a member of Ω. Now since the maximal such subsets X of Ω form a partition of Ω, for each BPHZ forest F, the pair (F, F) is a member of precisely on such maximal subset X of Ω. We then, bearing in mind that the definition of Ω refers to a particular position-space configuration x of the vertices of the original Feynman diagram, define, for each BPHZ forest F, the pair (P, Q) to whose sector the pair (F, x) belongs, to be the pair $(\tilde{P}, \tilde{Q})$, where $\tilde{P}$ is the intersection of the P's of all the members of the maximal such subset X of Ω of which (F, F) is a member, and $\tilde{Q}$ is the union of the Q's of all the members of the maximal such subset X of Ω of which (F, F) is a member. This gives a solution of the tesselation problem for which the answer to both questions 1) and 2) above is, "Yes". In practice, for any BPHZ forest F, the pair (P, Q) to whose sector the pair (F, x)

belongs, is given by the smallest subset P of F, and the largest "superset" Q of F, such that (P, Q) is a member of Ω: our construction guarantees, (provided σ is chosen such that $0 < \sigma \leq \frac{1}{8}$), that this determines P and Q uniquely, as $\tilde{P}$ and $\tilde{Q}$ as defined above. The characteristic funciton, in configuration space, of the sector defined by the pair (P, Q) of BPHZ forests, is constructed on pages 46 and 47 of [26].

In practice, two modifications have to be made to the definition of Ω given above. The first modification is simply to impose an absolute upper limit on the diameter $\mathbf{L}(F, A, x)$ of any cluster. This is due to the smooth long-distance cutoffs we impose on the propagators in our counterterms, and ensures that within any cluster, the propagators in the counterterm are identical to the original propagators. (The onset of the smooth long-distance cutoffs imposed on the propagators in the counterterms occurs at a value of $|x - y|$ strictly greater than zero.) The second modification involves restricting the propagators $\{i, j\}$ to which we apply (6.8) in the definition of Ω. For a given member A of Q that is not a member of P, instead of applying (6.8) for *all* propagators $\{i, j\}$ such that i is a member of A and j is not a member of A, (or in other words, for *all* legs of A), we apply (6.8) only for all propagators $\{i, j\}$ such that i is a member of A, and j is not a member of A, but *is* a member of every member of P that contains A as a subset. In other words, we apply (6.8) only to those legs of A that are *internal* lines of every member of P that contains A as a subset. The reason for this is that, in the course of the convergence proof, for the sector defined by the pair (P, Q) of BPHZ forests, we eventually bound the sum of the integrands for all the BPHZ forests F such that $P \subseteq F \subseteq Q$, by an expression that *factors* over the members of P, and has a form such that we can apply the Cluster Convergence Theorem [24] separately for the integral associated with each member of P. However with the original form of the definition of Ω, this factorization is spoilt by the occurrence of inequalities between the sizes of clusters in the factor associated with one member of P, and distances between vertices in the factors associated with larger members of P. The above modification to the definition of Ω removes this problem. However it turns out that with the modified definition of Ω, the size of the fixed real number σ that occurs in (6.8) has to be restricted slightly further to ensure that the tesselation problem is still solved. A sufficient restriction is $0 < \sigma \leq \frac{3}{25}$. For details we refer to [26].

The additional considerations necessary to demonstrate manifest BPHZ convergence in position space when only one-line *irreducible* counterterms are allowed, are

based on an identity given in Lemma 34 of [26], which expresses the sum over BPHZ forests that have only one-line *irreducible* members, as an alternating sum over quantities that also include forest that have one-line *reducible* members. The extra forests cancel out of the alternating sum, and each individual term in the alternating sum can be treated by an extension of the techniques used in the previous case. Part of the demonstration involves showing that if, given any power-counting divergent one-line-*reducible* subdiagram, we do a separate full BPHZ renormalization for each separate one-line-*irreducible* component of that subdiagram, (including forest with both one-line-irreducible and one-line-reducible members within each of those separate one-line-*irreducible* components), then the *overall* counterterm for this expression is finite, provided we define it as a limit from propagators that are *smoothly* regularized at extremely short distances - this is where we have to do the integrations by parts that we mentioned before. After sweeping excessive derivatives on the "key" propagators of this expression, "off the edge" of this overall counterterm by means of integrations by prts and translation smoothness, we are left with an absolutely convergent expression. For details we refer to Theorem 2 of [26].

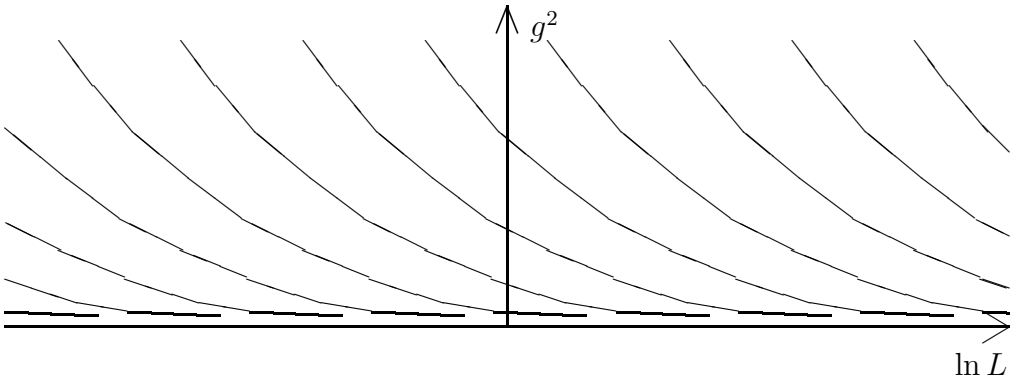
6.4 Renormalization group equation for a family of loops

We now briefly consider again the renormalization group equation (3.19) for a family of loops $W_{1L}, \ldots, W_{nL}$, where the loops $W_{1L}, \ldots, W_{nL}$ are obtained from a set of loops $W_1, \ldots, W_n$, such that the maximu distance between any two points on any of the loops $W_1, \ldots, W_n$ is R, (where R is the onset point of the smooth long-distance cutoffs we impose on the propagators in our BPHZ counterterms), by uniformly rescaling all the sizes and separations of the loops $W_1, \ldots, W_n$ until the maximum distance between any two points on any of them is L. (As usual in this paper we neglect the additional terms in (3.19) associated with the short-distance factors we divide our f_0's by.)

The "curves of constant f", i.e. of $\mathrm{d}f = 0$, in the (L, g) plane, are given by:

$$L\frac{\mathrm{d}g}{\mathrm{d}L} = \beta(g) \tag{6.11}$$

These curves look like:



(6.12)

On *these* "curves of constant f", g^2 *decreases* as L increases: the *opposite* behaviour to the "running coupling". But that is exactly right: to use this diagram, we suppose that $f_0(W_{1L}, \ldots, W_{nL}, g^2)$ has been calculated at one particular value of L, say L_0, for all g^2. Then to calculate $f_0(W_{1L}, \ldots, W_{nL}, g^2)$ for some particular value of g^2, say g_1^2, at some other value of L, say L_1, we identify the particular "curve of constant f" that passes through the point (L_1, g_1), and trace that curve back until it intersects the vertical line $L = L_0$. Then if g_0 is the value of g where this curve intersects the vertical line $L = L_0$, we have:

$$f_0(W_{1L_1}, \ldots, W_{nL_1}, g_1^2) = f_0(W_{1L_0}, \ldots, W_{nL_0}, g_0^2) \tag{6.13}$$

We see immediately from diagram (6.12) that g_0^2 will be greater than g_1^2 if L_1 is greater than L_0, and g_0^2 will be less than g_1^2 is L_1 is less than L_0. In fact on dimensional grounds, for fixed g_1, g_0 only depends on L_0 and L_1 through the ratio $\frac{L_0}{L_1}$. Hence $L_0 \left.\frac{\mathrm{d}g_0}{\mathrm{d}L_0}\right|_{g_1} = -L_1 \left.\frac{\mathrm{d}g_0}{\mathrm{d}L_1}\right|_{g_1}$, hence by (6.11):

$$L_1 \left.\frac{\mathrm{d}g_0}{\mathrm{d}L_1}\right|_{g_1} = -\beta(g_0) \tag{6.14}$$

Hence, by comparison with (5.30), we see that the dependence of g_0 on L_1, for fixed L_0 and g_1, is indeed given precisely by the "running coupling".

6.5 The contribution of the one-loop islands must have the *opposite* sign to what a scalar boson would give

We now consider the crucial question of the signs of the island diagrams, and of the sign of $\sum_i Y_i$ in equation (5.10), which, as we have noted, must be *negative*. Now from equation (4.74), we see that the coefficients $\frac{\mathrm{d}}{\mathrm{d}M}\mathbf{C}(M)\Big|_{M=1}$ tend to alternate in sign. However, as we explained in the discussion after equation (6.4), we expect that for input values of g^2 not greater than the critical value, and for reasonable ansätze for our long-distance factors (5.38), the sums over the *explicit* powers of g^2 that occur in the right-hand sides of the group-variation equations, and in the short-distance factors when we restore them in the windows of the right-hand side group-variation equation diagrams, will *converge*, and that the convergence rate will be geometric. It is thus highly unlikely that the sum over the contributions of all the island diagrams will have the opposite sign from the net contribution of the leading island diagrams, namely the one-loop islands. Therefore the net contribution of the one-loop island diagrams to $\sum_i Y_i$ in (5.10) must be *negative*. (Of course, since *every* island diagram has *exactly* one island, and *no* 45-paths that do *not* form part of that island, it makes no difference whether we say "one-loop islands" or "one-loop island diagrams".)

Let us consider carefully what determines the signs of the one-loop island diagrams. The factor $\frac{\mathrm{d}}{\mathrm{d}M}\mathbf{C}(M)\Big|_{M=1}$ is equal to 1. Now if, instead of Landau-gauge vector bosons, and Fadeev-Popov scalar fermions, we had a simple scalar boson, then the one-loop island would simply be $-\frac{1}{2}\mathrm{tr}\ln\left(\bar{D}^2\right)$, which we see from (1.33), the paragraph after (1.33), and (1.27), is simply a sum over closed paths, *all with positive coefficients*, with each path being weighted, by the 45-term in (2.6) and our general rules of right-hand side group-variation equation diagrams, by a product of two oppositely-directed fundamental representation Wilson loops, each in a *different* SU(N) group, (i.e. with a different set of SU(N) gauge fields). And if the island diagram is like(4.25) or (4.53), which in the discussion between equations (4.90) and (4.91), we called Type-1 island diagrams, then one of the two SU(N) groups occurs in no other path, so we have a Wilson-loop vacuum expectation value, (corresponding to the interior window of the island), for that SU(N) Wilson loop, while the other SU(N) group also occurs in the other connected components of the border of the non-simply connected window

surrounding the island, so that we have a Wilson-loop correlation function involving the border of the island, and the other connected components of the border of that non-simply connected window, (which are the loops that occur in the left-hand side f_0), for that SU(N) group. And if the island diagram is a Type-2 island diagram, for example (4.55), (4.71), or (4.72), then each of the two SU(N) fundamental representation Wilson loops weighting the island's 45-path is involved in a correlation function with one or more left-hand side loops.

Now these vacuum expectation values and correlation functions may all be expected to be predominantly *positive*, hence there is no doubt that if, instead of our Landau-gauge vector bosons and Fadeev-Popov scalar fermions, we had a simple scalar boson, then the contribution of the one-loop island would be *positive*: the *opposite* of what we need for (5.10) and the group-variation equations in general. Now (5.10) is certainly correct, (subject to the modification which we explain later, which has no effect on the sign). *We are forced to the conclusion that the contribution of the one-loop islands must have the* opposite *sign to what we would expect if, instead of our Landau-gauge vector bosons and Fadeev-Popov scalar fermions, we had a simple scalar boson.* Now (5.10) applies when our ansatz is valid, which requires all the sizes and separations of the left-hand side loops to be greater than $\frac{1}{\mu}$, but as we will see, the island diagrams also play a major role in driving the transition from asymptotic-freedom behaviour to the area law behaviour of our ansatz. in the *transition* domain, the most important island configurations are those where the island path closely tracks the left-hand side path, (in the group-variation equation for $f_0(W_1, g^2)$), and the window surrounding the island is a thin ribbon whose edges are the left-hand side loop and the border of the island.[3] In this domain we treat the *non-simply connected* window *by perturbation theory*, thus decoupling the group-variation equations for the multi-loop correlation functions, and obtaining, approximately, a simple linear differential equation for $f_0(W_{1L}, g^2)$. The conclusion with regard to the sign is as before: the sign of the contribution from the one-loop island diagrams must be *opposite* to what we would expect if, instead of our Landau-gauge vector bosons and Fadeev-Popov scalar fermions, we had a simple scalar boson. Our conclusion must be that the fact that $2g\beta(g)$ is negative, must be the first of a whole family of results, which guarantee that the net contribution of the

[3]Note added: this statement may be incorrect, but it may still be possible to investigate the transition region with the non-simply connected window treated in perturbation theory, thus obtaining an equation for the transition region which closes among the $f_0(W_1, g^2)$.

one-loop island diagrams to the right-hand sides of the group-variation equations in their standard forms (4.74), (4.76), etc., are negative under a rather wide range of conditions. Is this possible?

6.6 The required sign change occurs in the simplest example

Now from (1.35) and (1.42) we see that the contribution of the vector boson one-loop island is:

$$-2\mathrm{tr}\ln\left(\bar{D}^2\right)-\frac{1}{2}\mathrm{tr}\ln E+\frac{1}{2}\mathrm{tr}\left(G_0F\right)+\frac{1}{4}\mathrm{tr}\left(G_0FG_0F\right)+\frac{1}{6}\mathrm{tr}\left(G_0FG_0FG_0F\right)+\ldots \quad (6.15)$$

where g^2G_0 denotes the propagator matrix (1.20), F denotes the matrix $\begin{pmatrix} 2\bar{F}_{\mu\sigma} & 0 \\ 0 & 0 \end{pmatrix}$, the operator E is defined by (1.16), and $\ln E$ is to be expanded in powers of $(E-1)$.

The Fadeev-Popov contribution is $\mathrm{tr}\ln(\bar{D}^2)$, hence the total contribution of the one-loop islands is:

$$-\mathrm{tr}\ln\left(\bar{D}^2\right)-\frac{1}{2}\mathrm{tr}\ln E+\frac{1}{2}\mathrm{tr}\left(G_0F\right)+\frac{1}{4}\mathrm{tr}\left(G_0FG_0F\right)+\frac{1}{6}\mathrm{tr}\left(G_0FG_0FG_0F\right)+\ldots \quad (6.16)$$

This is to be interpreted in terms of window-weighted path integrals in the same way as we discussed above for a "scalar boson" one-loop island. Now the first term in (6.16) is simply twice the result for a "scalar boson" island, hence is expected to be *positive* for the same reasons as before. Hence the remaining terms must reverse the sign.

To consider whether this is possible, we have first to generalize (2.6) to cases where we have $\bar{D}_\mu$ or $\bar{F}_{\mu\nu}$ insertions along the paths, to deal with the $\mathrm{tr}\ln E$ and $\mathrm{tr}\left(G_0FG_0F\ldots G_0F\right)$ terms in (6.16). The required results may be obtained directly from (2.6), by making use of the $\sigma\to 0$ limit of (1.29), and the equation:

$$\bar{F}_{\mu\nu a}f_{AaB}=\left(\bar{D}_\mu\bar{D}_\nu-\bar{D}_\nu\bar{D}_\mu\right)_{AB} \quad (6.17)$$

which follows directly from (1.5), (1.4), and the definition of $\bar{F}_{\mu\nu a}$ after equation (1.11).

Now our path integrals have to be treated by the limiting procedure discussed in connection with expression (5.54) above. The $\bar{D}_\mu$ insertions affect both the long-distance factor and the short-distance factor. The effect on the long-distance factor is in general nonzero and finite. From (1.29) we see it essentially consists in the insertion

of *one* extra segment, of length ϵ say, into the sequence of a finite number of straight segments which defines the long-distance factor, and evaluating the derivative with respect to ϵ and $\epsilon = 0$. Only one straight segment of the original path need be affected: it is replaced by the new segment of length ϵ, plus a segment "nearly" the same as the original segment (as $\epsilon \to 0$), making the third side of a triangle with the original segment and the new segment of length ϵ. The change to the area of the minimal-area orientable spanning surface resulting from such a change to the path is in general of order ϵ, so we expect the effect of $\bar{D}_\mu$ insertions to get a finite contribution from the long-distance factor. We may expect this result to remain true if we take the limit in the derivative simultaneously with the limit $\sigma \to 0$, as suggested in (1.29), rather than taking the limit in the derivative with σ fixed. However the predominant effect of the $\bar{D}_\mu$ and $\bar{F}_{\mu\nu}$ insertions comes from the *short*-distance factor, which must be treated in the path integrals as discussed in connection with expression (5.54). Thus to determine the signs of the island diagrams we need the full renormalization of the group-variation equations, which is treated in our next paper. Here we shall simply show that the $\mathrm{tr} \ln E$ and $\mathrm{tr}\,(G_0 F G_0 F \ldots G_0 F)$ terms in (6.16) do indeed produce the required sign reversal in the simplest meaningful calculation, namely the change in the counterterm for the one-loop islands when we change the onset point of the smooth long-distance cutoffs imposed on the propagators in the counterterms, from R to R_2. Thus we neglect any effect of the $\bar{D}_\mu$ and $\bar{F}_{\mu\nu}$ insertions on the long-distance factors. We work through quadratic order in the effective fields introduced as in (5.49) to represent the short-distance factor. Now in general we have to introduce an independent set of $\mathrm{SU}(N)$ effective gauge fields for each window of our right-hand side group-variation equation diagrams: these correspond to different $\mathrm{SU}(N)$ subgroups of $\mathrm{SU}(NM)$ and $(\mathrm{SU}(N))^M$, as in (1.58). However through quadratic order the effects of the effective fields introduced in different windows decouple from one another: this is because the traces of all the $\mathrm{SU}(N)$ fundamental representation generators are zero, so we need either zero or two fields in each trace to get a nonzero result. Thus we consider explicitly the contributions to (6.16) quadratic in the effective fields $A_{\mu ax}$ introduced for one "side" of the closed 45-path that defines a one-loop island, and of zeroth order in the effective fields, (another set of $\mathrm{SU}(N)$ gauge fields, independent of $A_{\mu ax}$, introduced for the other "side" of that closed 45-path. We will extract the change in the logarithmically divergent part of the counterterm when we change the onset point of the smooth long-distance cutoffs imposed on the propagators in the counterterms, from R

to R_2. Now as explained after (5.14), the long-distance cutoffs on the propagators in the counterterms are imposed by multiplying them by a factor $f\left(\frac{(x-y)^2}{R^2}\right)$, where x and y are the positions of the ends of the propagator, and $f(s)$ is a fixed member of $\mathbf{R}^{\mathbf{R}}$ that is infinitely differentiable for all $s \in \mathbf{R}$, equal to 1 for $s \leq 1$, and equal to 0 for $s \geq t$, where t is a fixed finite real number strictly greater than 1. The consequence of this for the present calculation, where all the relevant "subdiagrams" have just two vertices, x and y, one of which is chosen as the contraction point, is that, after the angular integrations are done, the contribution to the logarithmically divergent part of the counterterm from the domain $|x-y| \geq R$ has the form:

$$\int_R^{R\sqrt{t}} \frac{\mathrm{d}u}{u} h\left(\frac{u}{R}\right) \tag{6.18}$$

for some smooth real function $h(s)$, and is thus completely independent of R. Thus if, for example, we assume $R_2 \geq R$, then the change to the logarithmically divergent part of the counterterm, when R is replaced by R_2, is given entirely by the domain $R \leq |x-y| \leq R_2$, calculated with the *original* propagators.

We now determine the contributions to this calculation of the separate terms in (6.16). We define the matrix Δ by:

$$\Delta_{xy} = \left(\frac{-1}{\partial^2}\right)_{xy} = \frac{1}{4\pi^2(x-y)^2} \tag{6.19}$$

Now

$$\bar{D}^2 = \partial^2 + A\partial + \partial A + AA \tag{6.20}$$

hence by (1.34) we have:

$$\operatorname{tr}\ln\left(\bar{D}^2\right) = \operatorname{tr}(\ln(\partial^2) + \ln(1 - \Delta(A\partial + \partial A + AA))) =$$

$$= \operatorname{tr}\left(\ln(\partial^2) - \Delta(A\partial + \partial A + AA) - \frac{1}{2}\Delta(A\partial + \partial A)\Delta(A\partial + \partial A) - \dots\right) \tag{6.21}$$

where the terms not explicitly displayed are of third degree or higher in $A_{\mu ax}$. Now the term $\operatorname{tr}\ln(\partial^2)$ is of no interest, and the trace on the group indices vanishes for the terms linear in $A_{\mu ax}$, hence for our calculation we may take

$$-\operatorname{tr}\ln(\bar{D}^2) = \operatorname{tr}(\Delta AA) + \frac{1}{2}\operatorname{tr}(\Delta(A\partial + \partial A)\Delta(A\partial + \partial A)) \tag{6.22}$$

But the counterterm for $\operatorname{tr}(\Delta AA)$ is completely independent of R, hence the only term in $-\operatorname{tr}\ln(\bar{D}^2)$ that contributes to our calculation is the term:

$$\frac{1}{2}\operatorname{tr}(\Delta(A\partial + \partial A)\Delta(A\partial + \partial A)) \tag{6.23}$$

This term gives

$$-\left(\Delta_{xy}\left(\hat{y}_\mu\hat{y}_\nu\Delta_{xy}\right)-\left(\hat{y}_\mu\Delta_{xy}\right)\left(\hat{y}_\nu\Delta_{xy}\right)\right)A_{\mu ax}A_{\nu ay} \tag{6.24}$$

where $\hat{y}_\mu$ means $\frac{\partial}{\partial y_\mu}$, etc., and the overall minus sign comes from (1.44), and is due to the fact that our representation matrices t_a are *anti*hermitian. Now (6.24) is quadratically divergent, so, choosing x as the contraction point, the logarithmically divergent part of the counterterm is given by taking the second-degree term $\frac{1}{2}(y-x)_\alpha(y-x)_\beta\hat{x}_\alpha\hat{x}_\beta A_{\nu ax}$ in the Taylor expansion of $A_{\nu ay}$ about the point $y=x$. Hence, defining $u\equiv y-x$, (6.24) gives:

$$\begin{aligned}
&-\int_{R\leq|u|\leq R_2}\mathrm{d}^4u\left(\frac{1}{4\pi^2u^2}\left(\frac{-2\delta_{\mu\nu}}{4\pi^2\left|u\right|^4}+\frac{8u_\mu u_\nu}{4\pi^2\left|u\right|^6}\right)-\frac{-2u_\mu}{4\pi^2\left|u\right|^4}\frac{-2u_\nu}{4\pi^2\left|u\right|^4}\right)A_{\mu ax}\frac{1}{2}u_\alpha u_\beta\hat{x}_\alpha\hat{x}_\beta A_{\nu ax}\\
&\quad=\frac{-1}{(4\pi^2)^2}\int_{R\leq|u|\leq R_2}\mathrm{d}^4u\left(\frac{-2\delta_{\mu\nu}u_\alpha u_\beta}{\left|u\right|^6}+\frac{4u_\mu u_\nu u_\alpha u_\beta}{\left|u\right|^8}\right)\frac{1}{2}A_{\mu ax}\hat{x}_\alpha\hat{x}_\beta A_{\nu ax}=\\
&=\frac{-1}{8\pi^2}\ln\left(\frac{R_2}{R}\right)\left(\frac{-2\delta_{\mu\nu}\delta_{\alpha\beta}}{4}+\frac{4\left(\delta_{\mu\nu}\delta_{\alpha\beta}+\delta_{\mu\alpha}\delta_{\nu\beta}+\delta_{\mu\beta}\delta_{\nu\alpha}\right)}{24}\right)\frac{1}{2}A_{\mu ax}\hat{x}_\alpha\hat{x}_\beta A_{\nu ax}\\
&\quad=\frac{-1}{96\pi^2}\ln\left(\frac{R_2}{R}\right)F_{\mu\nu ax}F_{\mu\nu ax}
\end{aligned}\tag{6.25}$$

where in the last line here we neglected a total derivative, and of course since we are neglecting terms of third degree and higher in $A_{\mu ax}$, $F_{\mu\nu ax}F_{\mu\nu ax}$ is equal to $\left(\hat{x}_\mu A_{\nu ax}-\hat{x}_\nu A_{\mu ax}\right)\left(\hat{x}_\mu A_{\nu ax}-\hat{x}_\nu A_{\mu ax}\right)$.

Now by (1.16), E is equal to $\bar{D}_\mu\frac{1}{D^2}\bar{D}_\nu$, and $\mathrm{tr}\ln E$, and the $\frac{1}{E}$ that occurs in G_0 by (1.20), are to be expanded in powers of $(E-1)$. Now:

$$\mathrm{tr}\ln E=\mathrm{tr}(E-1)-\frac{1}{2}\mathrm{tr}((E-1)^2)+\ldots \tag{6.26}$$

But $\mathrm{tr}E=1$, hence through quadratic order in the fields we have:

$$\mathrm{tr}\ln E=-\frac{1}{2}\mathrm{tr}((E-1)^2) \tag{6.27}$$

Now $(E-1)$ begins with a term linear in the fields, hence we see from (6.27) that to calculate $\mathrm{tr}\ln E$ through quadratic order in the fields we *only* need the term in $(E-1)$ *linear* in the fields. Furthermore, to calculate $\mathrm{tr}\left(G_0F\right)$ through quadratic order in the fields we again only need the term in $(E-1)$ linear in the fields, and to calculate $\mathrm{tr}\left(G_0FG_0F\right)$ through quadratic order in the fields we may set $E=1$. Thus for our calculation we only need the term in $(E-1)$ linear in the fields.

Now by (6.17), $[\bar{D}_\mu, \bar{D}_\nu] = \bar{F}_{\mu\nu}$, hence $[\bar{D}_\mu, \bar{D}^2] = \bar{F}_{\mu\nu}\bar{D}_\nu + \bar{D}_\nu\bar{F}_{\mu\nu}$, hence

$$[\bar{D}_\mu, \frac{1}{\bar{D}^2}] = -\frac{1}{\bar{D}^2}[\bar{D}_\mu, \bar{D}^2]\frac{1}{\bar{D}^2} = -\frac{1}{\bar{D}^2}\left(\bar{F}_{\mu\nu}\bar{D}_\nu + \bar{D}_\nu\bar{F}_{\mu\nu}\right)\frac{1}{\bar{D}^2} \tag{6.28}$$

hence

$$\begin{aligned} E - 1 = [\bar{D}_\mu, \frac{1}{\bar{D}^2}]\bar{D}_\mu = -\frac{1}{\bar{D}^2}\left(\bar{F}_{\mu\nu}\bar{D}_\nu + \bar{D}_\nu\bar{F}_{\mu\nu}\right)\frac{1}{\bar{D}^2}\bar{D}_\mu = \\ = -\Delta\left(\bar{F}_{\mu\nu}\partial_\nu + \partial_\nu\bar{F}_{\mu\nu}\right)\Delta\partial_\mu + \text{ quadratic in the fields} \\ = \partial_\mu\Delta\bar{F}_{\mu\nu}\Delta\partial_\nu + \text{ quadratic in the fields} \end{aligned} \tag{6.29}$$

Now (6.29 is based on (6.17), and when inserted in a 45-path, includes contributions from the effective fields introduced for *both* "sides" of that 45-path. However, as explained above, the contributions from the two sides of the closed 45-path that forms a one-loop island decouple through the quadratic order to which we are working, and we are calculating the contributions from *one* side of the 45-path. (If we were calculating the contributions from both sides of the path, we would have needed additional terms in (6.20), involving a second set of SU(N) gauge fields, and we would have had to display, in (6.20), the separate SU(N) fundamental representation indices for each side of the path.) Now the 45-term in (2.6) shows that, of the two SU(N) fundamental representation path-ordered phase factors along the 45-path, one is directed in the same direction as the 45-path, and the other is directed in the opposite direction to the 45-path, and we readily verify, from (6.17), (1.29), and (2.6), that if we neglect the effect of $\bar{D}_\mu$ and $\bar{D}_\nu$ on the long-distance factors, then the insertion of $\bar{F}_{\mu\nu}$ into a 45-path is equivalent to the insertion of $F_{\mu\nu}$ into the SU(N) fundamental representation path-ordered phase factor directed in the *same* direction as the 45-path, *minus* the insertion of $F_{\mu\nu}$ into the SU(N) fundamental representation path-ordered phase factor directed in the *opposite* direction to the 45-path, with the gauge fields in each inserted $F_{\mu\nu}$ being those of the phase factor into which the insertion occurs. The same result may be obtained directly from the formula for $\bar{D}_\mu$ as inserted into a 45-path:

$$\begin{aligned} \left(\bar{D}_\mu\right)_{jk,qp} = \delta_{jq}\delta_{kp}\partial_\mu + A_{1\mu a}\left(t_a\right)_{jq}\delta_{kp} + \delta_{jq}A_{2\mu a}\left(\left(t_a\right)_{kp}\right)^* = \\ = \delta_{jq}\delta_{pk}\partial_\mu + A_{1\mu a}\left(t_a\right)_{jq}\delta_{pk} - \delta_{jq}A_{2\mu a}\left(t_a\right)_{pk} \end{aligned} \tag{6.30}$$

where the A_1's and A_2's are two completely independent sets of SU(N) gauge fields, with the A_1's occurring in the SU(N) fundamental representation phase factor directed in the *same* direction as the 45-path, and the A_2's occurring in the SU(N) fundamental

representation phase factor directed in the *opposite* direction to the 45-path. Indeed, we find immediately from (6.30) that:

$$\left(\bar{D}_\mu\right)_{jk,qp}\left(\bar{D}_\nu\right)_{qp,sr} - \left(\bar{D}_\nu\right)_{jk,qp}\left(\bar{D}_\mu\right)_{qp,sr} =$$

$$= \Big\{ \left(\partial_\mu A_{1\nu a} - \partial_\nu A_{1\mu a} + f_{abc}A_{1\mu b}A_{1\nu c}\right)(t_1)_{js}\,\delta_{rk}$$

$$-\delta_{js}\left(\partial_\mu A_{2\nu a} - \partial_\nu A_{2\mu a} + f_{abc}A_{2\mu b}A_{2\nu c}\right)(t_a)_{rk}\Big\} =$$

$$= F_{1\mu\nu a}\,(t_a)_{js}\,\delta_{rk} - \delta_{js}F_{2\mu\nu a}\,(t_a)_{rk} \tag{6.31}$$

Hence, bearing in mind that, as explained, the effects of the two sides of the 45-path decouple through the quadratic order to which we are working, we will complete this calculation for the "side" of the 45-path directed in the *same* direction as the 45-path itself, and accordingly replace $\bar{F}_{\mu\nu}$ in (6.29) by $F_{1\mu\nu} = F_{1\mu\nu a}\,(t_a)_{ij}$. Hence from (6.27) and (6.29) we find that the contribution to the $-\frac{1}{2}\mathrm{tr}\ln E$ term in (6.16) from "our" side of the 45-path is:

$$\frac{1}{4}\mathrm{tr}\left(\partial_\mu\Delta F_{1\mu\nu}\Delta\partial_\nu\partial_\sigma\Delta F_{1\sigma\tau}\Delta\partial_\tau\right) \tag{6.32}$$

Now Δ_{xy} satisfies the identity:

$$\left(\Delta\partial_\mu\partial_\nu\Delta\right)_{xy} = \frac{-\delta_{\mu\nu}}{8\pi^2(x-y)^2} + \frac{(x-y)_\mu(x-y)_\nu}{4\pi^2\left|x-y\right|^4} \tag{6.33}$$

hence (6.32 is equal to:

$$-\frac{1}{4}\int\int \mathrm{d}^4x\mathrm{d}^4y\left(\frac{-\delta_{\nu\sigma}}{8\pi^2u^2} + \frac{u_\nu u_\sigma}{4\pi^2\left|u\right|^4}\right)\left(\frac{-\delta_{\tau\mu}}{8\pi^2u^2} + \frac{u_\tau u_\mu}{4\pi^2\left|u\right|^4}\right)F_{1\mu\nu ax}F_{1\sigma\tau ay} \tag{6.34}$$

where the overall minus sign again comes from (1.44), and we have defined $u \equiv y - x$ as before. Now (6.34) is logarithmically divergent, hence, again choosing x as the contraction point, its contribution to our calculation is:

$$\frac{-1}{4(4\pi^2)^2}\int_{R\leq|u|\leq R_2}\mathrm{d}^4u\left(\frac{\delta_{\nu\sigma}\delta_{\tau\mu}}{4\left|u\right|^4} - \frac{\delta_{\nu\sigma}u_\tau u_\mu + \delta_{\tau\mu}u_\nu u_\sigma}{2\left|u\right|^6} + \frac{u_\nu u_\sigma u_\tau u_\mu}{\left|u\right|^8}\right)F_{1\mu\nu ax}F_{1\sigma\tau ax}$$

$$= \frac{-1}{32\pi^2}\ln\left(\frac{R_2}{R}\right)\left(\delta_{\nu\sigma}\delta_{\tau\mu}\left(\frac{1}{4}-\frac{1}{8}-\frac{1}{8}\right) + \frac{\delta_{\nu\sigma}\delta_{\tau\mu} + \delta_{\nu\tau}\delta_{\sigma\nu} + \delta_{\nu\mu}\delta_{\sigma\tau}}{24}\right)F_{1\mu\nu ax}F_{1\sigma\tau ax}$$

$$= 0 \tag{6.35}$$

We next determine the contribution of the term $\frac{1}{2}\mathrm{tr}\left(G_0F\right)$ in (6.16). Now by (1.20) we have:

$$\mathrm{tr}\left(G_0F\right) = \mathrm{tr}\left(\left(\frac{-1}{\bar{D}^2}\delta_{\mu\nu} + \frac{1}{\bar{D}^2}\bar{D}^\mu\frac{1}{E}\bar{D}_\nu\frac{1}{\bar{D}^2}\right)2\bar{F}_{\nu\sigma}\right) =$$

$$\mathrm{tr}\left(\frac{1}{\bar{D}^2}\bar{D}_\mu\frac{1}{E}\bar{D}_\nu\frac{1}{\bar{D}^2}2\bar{F}_{\nu\mu}\right) =$$
$$= -\mathrm{tr}\left(\Delta\bar{F}_{\mu\nu}\Delta\bar{F}_{\mu\nu}\right) - \mathrm{tr}\left(\Delta\partial_\mu\partial_\sigma\Delta\bar{F}_{\sigma\tau}\Delta\partial_\tau\partial_\nu\Delta 2\bar{F}_{\mu\nu}\right) + \text{ cubic} \tag{6.36}$$

where we used (6.17) and (6.29). We again extract the contribution of *one* side of the 45-path by replacing $\bar{F}_{\mu\nu}$ by $F_{1\mu\nu}$, and note that the second term in (6.36) has the same form as (6.32), so that by (6.35) it gives no contribution to our calculation. The first term in (6.36) is logarithmically divergent, hence with the same conventions as before, the contribution to our calculation of the $\frac{1}{2}\mathrm{tr}\,(G_0F)$ term in (6.16) is:

$$\frac{1}{2(4\pi^2)^2}\int_{R\le|u|\le R_2}\frac{\mathrm{d}^4u}{|u|^4}F_{1\mu\nu ax}F_{1\mu\nu ax} = \frac{1}{16\pi^2}\ln\left(\frac{R_2}{R}\right)F_{1\mu\nu ax}F_{1\mu\nu ax} \tag{6.37}$$

where the overall sign is again due to (1.44).

Finally we determine the contribution of the $\frac{1}{4}\mathrm{tr}\,(G_0FG_0F)$ term in (6.16). From (1.20) we find:

$$\mathrm{tr}\,(G_0FG_0F) = 4\mathrm{tr}\left(\left(\Delta\delta_{\mu\nu}+\Delta\partial_\mu\partial_\nu\Delta\right)\bar{F}_{\nu\sigma}\left(\Delta\delta_{\sigma\tau}+\Delta\partial_\sigma\partial_\tau\Delta\right)\bar{F}_{\tau\mu}\right) + \text{ cubic} \tag{6.38}$$

We again extract the contribution of *one* side of the 45-path by replacing $\bar{F}_{\mu\nu}$ by $F_{1\mu\nu}$. Expression (6.38) is again logarithmically divergent, hence by (6.19) and (6.33), and with the same conventions as before, the contribution to our calculation of the $\frac{1}{4}\mathrm{tr}\,(G_0FG_0F)$ term in (6.16) is:

$$-\int_{R\le|u|\le R_2}\mathrm{d}^4u\left(\frac{\delta_{\mu\nu}}{8\pi^2u^2}+\frac{u_\mu u_\nu}{4\pi^2\,|u|^4}\right)\left(\frac{\delta_{\sigma\tau}}{8\pi^2u^2}+\frac{u_\sigma u_\tau}{4\pi^2\,|u|^4}\right)F_{1\nu\sigma ax}F_{1\tau\mu ax} =$$
$$= \frac{-1}{8\pi^2}\ln\left(\frac{R_2}{R}\right)\left(\delta_{\mu\nu}\delta_{\sigma\tau}\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{8}\right)+\frac{\delta_{\mu\nu}\delta_{\sigma\tau}+\delta_{\mu\sigma}\delta_{\nu\tau}+\delta_{\mu\tau}\delta_{\nu\sigma}}{24}\right)F_{1\nu\sigma ax}F_{1\tau\mu ax}$$
$$= \frac{1}{16\pi^2}\ln\left(\frac{R_2}{R}\right)F_{1\mu\nu ax}F_{1\mu\nu ax} \tag{6.39}$$

where the overall sign is again due to (1.44).

Hence, bearing in mind that the result (6.25) is the contribution of *one* side of the 45-path for the term $-\mathrm{tr}\ln(\bar{D}^2)$ in (6.16), we find from (6.25), (6.35), (6.37), and (6.39), that through quadratic order in the effective fields for *one* side of the 45-path, the change in the counterterm for the one-loop islands, when we change the onset point of the smooth long-distance cutoffs imposed on the propagators in the counterterms, from R to R_2, is:

$$(-1+6+6)\frac{1}{96\pi^2}\ln\left(\frac{R_2}{R}\right)F_{1\mu\nu ax}F_{1\mu\nu ax} = \frac{11}{96\pi^2}\ln\left(\frac{R_2}{R}\right)F_{1\mu\nu ax}F_{1\mu\nu ax} \tag{6.40}$$

Thus the required sign change has indeed occurred.

6.7 The sign reversal is a consequence of the sign of the β-function

Now in fact the calculation we have just done precisely parallels a calculation of the leading β-function coefficient by the "gauge-covariant background field method" [62]. That method is based on the general result that the effective action Γ, (i.e. the generating functional of the proper vertices), is equal to minus the sum of all the connected, one-line-irreducible vacuum bubbles, calculated with an action that is obtained from the usual action $A(\phi)$, (where ϕ denotes al the fields we functionally integrate over), by replacing ϕ by $\Phi + \phi$, (where Φ represents all the "classical" fields, or "background" fields, and is the argument of Γ), and deleting the term linear in ϕ in the power series expansion of $A(\Phi + \phi)$ about $\phi = 0$. In other words, $\Gamma\Phi$ is equal to minus the sum of all the connected, one-line-irreducible vacuum bubbles, calculated with the action:

$$A(\Phi + \phi) - \phi_i \frac{\delta A(\Phi)}{\delta \Phi_i} \tag{6.41}$$

We are here using the same schematic notation that we used in (5.25) and (5.26). The "free" Φ propagator, in the presence of the "background fields" Φ, is given by the matrix N_{1ij} of (5.25). The cubic vertex given by the action (6.41) is $-\frac{\delta^3 A(\Phi)}{\delta\Phi_i \delta\Phi_j \delta\Phi_k}$, the quartic vertex is $-\frac{\delta^4 A(\Phi)}{\delta\Phi_i \delta\Phi_j \delta\Phi_k \delta\Phi_l}$, and so on, just as in the development of the matrix N, as illustrated in (5.26). For the one-loop effective action, the only term in (6.41) needed is the term:

$$\frac{1}{2}\phi_i \phi_j \frac{\delta^2 A(\Phi)}{\delta\Phi_i \delta\Phi_j} \tag{6.42}$$

The one-loop effective action is half the trace of the logarithm of the matrix $\frac{\delta^2 A(\Phi)}{\delta\Phi_i \delta\Phi_j}$. (The effective action is the *negative* of the sum of the vacuum bubbles due to our conventions (5.15) and (5.17).)

In the "gauge-covariant background field method", the above general results are distorted by modifying the gauge-fixing action so that the full quantum action, including the gauge-fixing and Fadeev-Popov terms, has exactly the same gauge-invariance in terms of the "classical" gauge field, say $\bar{A}_{\mu ax}$, as the original action had in terms of $A_{\mu ax}$. In other words, if $\bar{A}_{\mu ax}$ denotes the "classical" gauge field, (that becomes an argument of Γ), and $A_{\mu ax}$ denotes the "quantum" gauge field, (that we functionally integrate over), then where, in the gauge-fixing term, we would take $\partial_\mu \left(\bar{A}_{\mu ax} + A_{\mu ax} \right)$ by the above prescription, the "gauge-covariant background field method" instead takes

$\partial_\mu A_{\mu a x} + \bar{A}_{\mu b x} f_{abc} A_{\mu c x}$. The result of this is that the "gauge-covariant background field method" calculates no the true effective action $\Gamma(\bar{A}, \bar{B}, \bar{\phi}, \bar{\psi})$, but rather a different quantity $\tilde{\Gamma}(\bar{A})$. This quantity $\tilde{\Gamma}(\bar{A})$ has exactly the same gauge-invariance in terms of $\bar{A}_{\mu a x}$ as the original action has in terms of $A_{\mu a x}$, whereas the true effective action $\Gamma(\bar{A}, \bar{B}, \bar{\phi}, \bar{\psi})$ is *not* gauge-invariant, but instead satisfies the Ward identity (5.22). However the fact that $\tilde{\Gamma}(\bar{A})$ has exactly the same gauge-invariance in terms of $\bar{A}_{\mu a x}$ as the original action has in terms of $A_{\mu a x}$, means that, in our BPHZ approach, and with our conventions as in (1.51) and (5.14), the change in the one-loop counterterm for $\tilde{\Gamma}(\bar{A})$, when we change the onset point of the smooth long-distance cutoffs imposed on the propagators in the counterterm from R to R_2, is equal to $\frac{N}{4}$, times the coefficient of g^4 in $2g\beta(g)$, times $\ln\left(\frac{R_2}{R}\right)$, times $\bar{F}_{\mu\nu a x}\bar{F}_{\mu\nu a x}$, where

$$\bar{F}_{\mu\nu a x} = \partial_\mu \bar{A}_{\nu a x} - \partial_\nu \bar{A}_{\mu a x} + f_{abc}\bar{A}_{\mu b}\bar{A}_{\nu c} \tag{6.43}$$

The relevance of this to the group-variation equations is that there is a precise correspondence between expression (1.12), which in the context of the group-changing equations is the sum of all the terms in $F_{\mu\nu a}F_{\mu\nu a} + F_{\mu\nu A}F_{\mu\nu A}$ that contain exactly two A_μ's with upper-case group indices, and the expression:

$$2\left(\left(\bar{D}_\mu A_\nu\right)_a\left(\bar{D}_\mu A_\nu\right)_a - \left(\bar{D}_\nu A_\mu\right)_a\left(\bar{D}_\mu A_\nu\right)_a + \bar{F}_{\mu\nu a} f_{abc} A_{\mu b} A_{\nu c}\right) \tag{6.44}$$

where $\left(\bar{D}_\mu A_\nu\right)_a \equiv \partial_\mu A_{\nu a} + \bar{A}_{\mu b} f_{abc} A_{\nu c}$, and $\bar{F}_{\mu\nu a}$ is defined by (6.43), which in the context of the "gauge-covariant background field method" is the sum of all the terms in $F_{\mu\nu a}(\bar{A} + A)\, F_{\mu\nu a}(\bar{A} + A)$ that contain exactly two $A_{\mu a}$'s. Indeed, if in (1.12) we first put a bar above all the A_μ's with *lower*-case group indices, (which occur in $\bar{D}_\mu$ by (1.5) and in $\bar{F}_{\mu\nu a}$ by the definition after (1.11)), and then replace all upper-case group indices by lower-case group indices, we precisely obtain (6.44). Furthermore, the same treatment applied to the third and fourth terms in the gauge-fixing action (1.6) changes them precisely into the gauge-fixing terms for $A_{\mu a}$ in the "gauge-covariant background field method" when a gauge-fixing auxiliary field B_a is used in fixing the gauge of $A_{\mu a}$ in that method. Furthermore, if we set $\beta = 0$ in (1.6), as we of course do in the context of the group-changing equations and group-variation equations, then all the terms in the group-changing Fadeev-Popov action that contain upper-case indices and contribute to expectation values and correlation functions that contain no explicit Fadeev-Popov fields or gauge-fixing auxiliary fields, may be put in the form (1.23). And if we apply the same procedure as before to the first term in (1.23), namely putting a

bar over the $A_{\mu a}$ that occurs in $\bar{D}_\mu$, then replacing all upper-case indices by lower-case indices, we precisely obtain the part of the Fadeev-Popov action in the "gauge-covariant background field method" that is quadratic in the "quantum fields".

The consequence of this is that, as stated above, our calculation, through quadratic order in the effective fields for one side of the 45-path, of the change in the counterterm for the one-loop islands, when we change the onset point of the smooth long-distance cutoffs imposed on the propagators in the counterterms from R to R_2, precisely parallels the calculation of the change in the one-loop counterterm for $\tilde{\Gamma}(\bar{A})$, when we change the onset point of the smooth long-distance cutoffs imposed on the propagators in the counterterms from R to R_2. In fact, to obtain the corresponding change in the one-loop counterterm for $\tilde{\Gamma}(\bar{A})$ from (6.40), we multiply by -1, (since the effective action is the *negative* of the sum of the vacuum bubbles), replace $F_{1\mu\nu ax}F_{1\mu\nu ax}$ by $\bar{F}_{\mu\nu ax}\bar{F}_{\mu\nu ax}$, multiply by 2, (due to the two "sides" of an adjoint representation path in terms of fundamental representation paths), and multiply by N, (for the fundamental representation trace on the other "side" of the adjoint representation path.) This indeed gives the coefficient of g^4 in $2g\beta(g)$ in (5.35) by the relation stated above. This corresponds to the fact that, to one-loop order, the full action $S(A, B, \phi, \psi, g^2, \alpha) + C(A, B, \phi, \psi, g^2, \alpha, R)$, where $S(A, B, \phi, \psi, g^2, \alpha)$ is the seed action (5.14), and $C(A, B, \phi, \psi, g^2, \alpha, R)$ is the sum of all the counterterms, is left unaltered, if we *simultaneously* replace R by R_2 and g^2 by g_2^2, where:

$$\frac{N}{4g_2^2} + \frac{11N}{48\pi^2}\ln\left(\frac{R_2}{R}\right) = \frac{N}{4g^2} \tag{6.45}$$

and make the required finite rescalings of the fields, which are not detected by the "gauge-covariant background field method". We immediately obtain (5.35) from (6.45) by use of (5.34).

Thus the fact that the required sign change occurred in (6.40) is not an accident: it is an immediate corollary of the fact that $2g\beta(g)$ is negative. And we see furthermore that, as noted before (6.15), the fact that $2g\beta(g)$ is negative must, in the context of the group-variation equations, be the first of a whole family of results, which guarantee that the net contribution of the one-loop island diagrams to the right-hand sides of the group-variation equations in their standard forms (4.74), (4.76), etc., are negative under a rather wide range of conditions.

Chapter 7

Verification And Correction Of The Ansatz, And Zeroth Order Value Of $m_{0^{++}}/\sqrt{\sigma}$, If The Pre-Exponential Factor In The Cylinder-Topology Term Is Non-Marginal

We now leave further investigation of the signs of the island diagrams to our next paper, and, *assuming* they come out right, (i.e. negative), consider in more detail the result of substituting our ansatz for $f_0(W_1, \ldots, W_n, g^2)$, when the sizes and separations of $W_1, \ldots, W_n$ are all $\geq \frac{1}{\mu}$, as described after (5.1) above, into the right-hand sides of the group-variation equations. Our purpose is to determine whether our ansatz solves the group-variation equations when the sizes and separations of $W_1, \ldots, W_n$ are all $\geq \frac{1}{\mu}$.

7.1 The ansatz satisfies the Group-Variation Equations for the correlation functions

We begin by recalling that, in the discussion between (4.90) and (4.91), we defined an island diagram to be a Type-1 island diagram if, among all the windows beside paths of the island, exactly *one* of those windows is *not* simply connected, and we defined all other island diagrams to be Type-2 island diagrams. We note that *all* the island

diagrams in the right-hand side of the group-variation equation (4.74) for $f_0(W_1, g^2)$ are Type-1 island diagrams, and we also note that, among all the windows beside paths of the island in a Type-2 island diagram, at least two of those windows are *not* simply connected.

We recall from the start of this section that the principal effect of the window weights on the right-hand side group-variation diagrams is that the path sum for each 45-path gives approximately the free propagator for a *massive* scalar particle, and that , by the paragraph after (4.90), a conservative estimate of the effective mass is $\mu\sqrt{\frac{\pi}{2}} = 1.3\mu$, where μ^2 is the coefficient of the area in the Wilson area law in our ansatz. We have to note, however, that this estimate has taken no account of the details of the path sums for our Landau-gauge vector bosons, specifically the $F_{\mu\nu}$ insertions and the $\frac{1}{E}$ term in G_0, which, as we know, have to reverse the sign of the one-loop island diagrams in comparison to what we would get with a simple scalar boson, and which indeed do reverse the sign in the example considered above. We nevertheless assume that these effects do not reduce the effective mass of a 45-path below 1.3μ.

We begin by asking whether substituting our ansatz into the right-hand sides of the group-variation equations, correctly reproduces the factor $\sqrt{\frac{m}{32\pi^3 L^3}}e^{-mL}$ for each separate straight line of length L in our minimal-length spanning tree of $S_1, \ldots, S_p$, where $S_1, \ldots, S_p$ are the separate connected components of our absolute minimal-area orientable spanning surface of $W_1, \ldots, W_n$, and m is the mass of the lightest glueball, as in (5.4). To investigate this, we consider loops $W_1, \ldots, W_n$, $n \geq 2$, such that the diameter of $W_1, \ldots, W_n$, (i.e. the maximum distance between any two points on $W_1, \ldots, W_n$), is R, (where R is the onset point of the smooth long-distance cutoffs imposed on the propagators in our counterterms), and such that the *separations* between the loops $W_1, \ldots, W_n$ are all much larger than any of the *sizes* of the loops $W_1, \ldots, W_n$. We define $W_{1L}, \ldots, W_{nL}$ to be the loops obtained from $W_1, \ldots, W_n$ by uniformly scaling all the sizes and relative positions of $W_1, \ldots, W_n$ by the scale factor $\frac{L}{R}$. We now, as before, (in (5.6)), multiply both sides of the group-variation equation for $f_0(W_{1L} \ldots, W_{nL}, g^2)$ by $-\frac{2g\beta(g)}{g^2}$, after which, by (3.19), (and, as usual in this paper, neglecting the terms in the renormalization group equation (3.19) associated with the short-distance factors), the term $g^2\frac{\mathrm{d}}{\mathrm{d}g^2}f_0(W_{1L}, \ldots, W_{nL}, g^2)$ in the left-hand side of the group-variation equation for $f_0(W_{1L}, \ldots, W_{nL}, g^2)$ becomes simply:

$$L\frac{\partial}{\partial L}f_0(W_{1L}, \ldots, W_{nL}, g^2) \tag{7.1}$$

Now by our assumption that the *separations* between the loops $W_1, \ldots, W_n$ are all much larger than their *sizes*, the absolute minimal-area orientable spanning surface S_L of $W_{1L}, \ldots, W_{nL}$ will have a separate connected component S_{iL} for each loop W_{iL}, which wil be the minimal-area orientable spanning surface of W_{iL}, of area $a_i L^2$, say. Let the lengths of the separate straight lines forming the minimal-length spanning tree of $S_{1L}, \ldots, S_{nL}$ be $b_1 L, \ldots, b_q L$, where $(n-1) \leq q \leq (2n-3)$. We assume that the a_i's have been chosen sufficiently small, and the separations between the W_i's sufficiently large, that there is a large range of L where our ansatz applies, and $(a_1 + \ldots + a_n)L^2$ is small compared to $(b_1 + \ldots + b_q)L$. Then in this range of L, the dominant term in the left-hand side of the group-variation equation for $f_0(W_{1L}, \ldots, W_{nL}, g^2)$, after multiplication by $-\frac{2\beta(g)}{g}$, comes from (6.45) acting on the exponents $e^{-mb_j L}$ in (5.4), and gives:

$$-m\,(b_1 + \ldots + b_q)\,L f_0(W_{1L}, \ldots, W_{nL}, g^2) \tag{7.2}$$

Now, what diagrams will give the predominant contributions to the right-hand side of the group-variation equation for $f_0(W_{1L}, \ldots, W_{nL}, g^2)$ in this situation? We see immediately that the answer depends on the actual value of the ratio $\frac{m}{\mu}$. Indeed, if m is less than twice the effective mass 1.3μ of the 45-paths, then only island diagrams can contribute significantly, while if m is greater than twice 1.3μ, then diagrams such as (4.49) and (4.68) will give the largest contributions to the right-hand side. But m *cannot* be strictly larger than twice the effective mass of the 45-paths, because if that were the case, then the left-hand side of the group-variation equation for $f_0(W_{1L}, \ldots, W_{nL}, g^2)$ would have a more rapid exponential fall-off with increasing L than the leading terms in the right-hand side, (unless the leading terms in the right-hand side somehow cancelled, which there is no reason to expect, and which we assume does *not* occur). Can m be *equal* to twice the effective mass of the 45-paths, or, more precisely, equal to the lowest-mass state of a cylinder with two 45-paths running along it? If this were the case, we would have to consider diagrams such as (4.49) and (4.68), but we readily see that, if the spectrum of the cylinder with two 45-paths along it is discrete, (as it must be), with lowest mass equal to m, then these diagrams must have the same L-dependence as our ansatz, and hence be smaller, by one power of L, than the leading term in the left-hand sides of the relevant group-variation equations, as given by (7.2). Thus, *if* our ansatz satisfies the group-variation equations in this region, then the leading terms in the right-hand sides must come from the island diagrams, irrespective of whether m is strictly less than, or equal to, the lowest-mass state of a cylinder with

two 45-paths along it. However, these two cases differ with respect to the particular types of island configurations that will give the leading contributions: if m is strictly less than the lowest-mass state of a cylinder with two 45-paths along it, then the dominant configurations will come from islands of size $\frac{1}{\mu}$, with the contributions of larger islands being suppressed exponentially, while if m is equal to the lowest-mass state of a cylinder with two 45-paths along it, then important contributions will also come from islands "stretched" along the spanning tree.

Let us first assume that m is strictly less than the lowest-mass state of a cylinder with two 45-paths along it, so that the dominant contributions will come from islands of L-independent size $\frac{1}{\mu}$. Which island diagrams will give the main contributions? Let us first consider the Type-1 island diagrams. Now in a Type-1 island diagram in the right-hand side of the group-variation equation for $f_0(W_{1L}, \ldots, W_{nL}, g^2)$, there is *one* window whose boundary has $(n+1)$ connected components, namely $W_{1L}, \ldots, W_{nL}$, and the outer boundary of the island, and all the other windows are simply-connected, and lie "inside" the island. Let $\tilde{W}$ denote the outer boundary of the island. Then the window weight for the non-simply connected window is $f_0(W_{1L}, \ldots, W_{nL}, \tilde{W}, g^2)$. Now, since we are assuming that m is strictly less than the lowest mass state of the cylinder with two 45-paths along it, the predominant contributions will come from $\tilde{W}$ with the L-independent size $\frac{1}{\mu}$. Thus, provided $\tilde{W}$ is not right next to one of the W_{iL}'s, the absolute minimal-area orientable spanning surface $\tilde{S}$ of $W_{1L}, \ldots, W_{nL}, \tilde{W}$, will have $(n+1)$ connected components, namely the absolute minimal-area orientable spanning surfaces of the individual loops W_{iL} and $\tilde{W}$. Now clearly the minimum possible total length of the straight line segments forming the minimal-length spanning tree of the $(n+1)$ connected components of $\tilde{S}$ will be attained if $\tilde{W}$ is situated such that one of the straight line segments of the minimal-length spanning tree of $S_{1L}, \ldots, S_{nL}$, where S_{iL} is the absolute minimal-area orientable spanning surface of W_{iL}, passes through the absolute minimal-area orientable spanning surface of $\tilde{W}$. And clearly the main contributions to this island diagram will come from configurations where $\tilde{W}$, with L-independent size $\frac{1}{\mu}$, which is small compared to any of the lengths $b_j L$ of the straight-line segments of the minimal-length spanning tree T_L of $S_{1L}, \ldots, S_{nL}$, is close to one of the straight-line segments of T_L. Now in any such configuration, if z denotes any point of the absolute minimal-area orientable spanning surface of $\tilde{W}$, and x and y denote the ends of the straight-line segment of T_L that $\tilde{W}$ is close to, then the angle $\widehat{xzy}$ will be much larger than $\frac{2\pi}{3}$, and in fact approaching π, hence by the discussion after (4.110),

the minimal-length spanning tree of x, y, and z, will consist of the two straight lines $\overline{xz}$ and $\overline{yz}$. Thus the minimal-length spanning tree $\tilde{T}$ of the $(n+1)$ connected components of $\tilde{S}$ will consist of all the straight segments of the minimal-length spanning tree T_L of $S_{1L}, \ldots, S_{nL}$ *except* the segment $\overline{xy}$, plus a straight line from x to some point z_1 on the absolute minimal-area orientable spanning surface of $\tilde{W}$, plus a straight line from y to some point z_2 on the absolute minimal-area orientable spanning surface of $\tilde{W}$.

Let us now, for simplicity, assume that m is sufficiently smaller than the lowest-mass state M of the cylinder with two 45-paths along it, (which we estimate roughly as twice the effective mass 1.3μ of the 45-paths), that there is no significant tendency for the island to "stretch" along the segment $\overline{xy}$ of the minimal-length spanning tree of $S_{1L}, \ldots, S_{nL}$ that it is close to. Then we may, to a good enough approximation, set $z_1 = z_2 = z$, where z is the centre of the island, i.e. the mean position of the vertices of the island. (If, on the other hand, $M - m$ was small compared to μ, we would have to allow the island to "stretch" to the L-independent length $\frac{1}{M-m}$ along the segment $\overline{xy}$, with z_1 and the end nearer x, and z_2 at the end nearer y, but for simplicity we assume that this is not the case.) We now see that, according to our ansatz, as described after (5.3), the contribution of this configuration to this island diagram, is equal to our ansatz for $f_0(W_{1L}, \ldots, W_{nL}, g^2)$, with the following changes: firstly, the factor $\sqrt{\frac{m}{32\pi^3|x-y|^3}}e^{-m|x-y|}$ associated with the straight line segment $\overline{xy}$ by point (ii) in our ansatz is replaced by the factor:

$$\sqrt{\frac{m}{32\pi^3\left|x-z\right|^3}}e^{-m|x-z|}\sqrt{\frac{m}{32\pi^3\left|y-z\right|^3}}e^{-m|y-z|} \tag{7.3}$$

secondly, by point (i) in our ansatz as applied to $f_0(W_{1L}, \ldots, W_{nL}, g^2)$, we get a factor $e^{-\mu^2\tilde{\tilde{A}}}$, where $\tilde{\tilde{A}}$ is the area of the absolute minimal-area orientable spanning surface $\tilde{\tilde{S}}$ of $\tilde{W}$, thirdly, by point (i) or our ansatz as applied, for each "internal" window U_i of the island, to $f_0(U_i, g^2)$, a factor $e^{-\mu^2 B_i}$, where B_i is the area of the absolute minimal-area orientable spanning surface of U_i, and fourthly, from point (iii) in our ansatz applied to the ends on $\tilde{\tilde{S}}$ of the line segments $\overline{xz}$ and $\overline{yz}$, a factor f^2, where f represents the coupling of the lightest glueball to a minimal-area orientable spanning surface.

We now see that, due to our assumption, which we made for simplicity, that m is sufficiently smaller than the mass M of the lowest-mass state of the cylinder with two 45-paths along it, that there is no significant tendency for the island to "stretch" along the segment $\overline{xy}$, (in consequence of which we may, in this preliminary analysis, set $z_1 = z_2 = z$, where z is the mean position of the vertices of the island), if we do

the path integrals for all the 45-paths in the island, subject to the mean position z of the vertices in all the 45-paths being fixed, then by the translation-invariance of the extra factors associated with $\tilde{\tilde{S}}$ and the internal windows of the island, the *only* z-dependence of the result is through the factor (7.3).

Now every Type-1 island is obtained from some connected, one-line irreducible vacuum bubble formed of 45-paths, that may be drawn on the surface of the 2-sphere without any 45-paths crossing one another, by cutting n holes in *one* of the windows of that vacuum bubble, and stretching that window to form the non-simply connected window that "surrounds" the island. Each such vacuum bubble gives rise to as many Type-1 island diagrams as it has windows. If the vacuum bubble has some symmetries, it will have associated with it a symmetry factor given by the reciprocal of the number of elements in the finite symmetry group under which the bubble is invariant, and we will also find that different choices of the window we put the n holes in lead to the same island diagram. In that case, the factor given by the number of windows that give the same island diagram, partly cancels the symmetry factor of the bubble, and any symmetry factor that remains is associated with a rotational symmetry of the island, as discussed in the paragraphs after (3.21). For example, the one-loop vector boson vacuum bubble has a symmetry factor $\frac{1}{2}$, due to the two-fold symmetry that rotates the two windows of the bubble into one another. This cancels the factor of 2 that arises because we get the *same* Type-1 island diagram irrespective of which of the two windows we put the n holes into. (We did not include this factor of 2 in our study above of the counterterm for the one-loop islands. The Fadeev-Popov one-loop vacuum bubble has no symmetry factor, and the two possible choices of which of the two windows we put the n holes in, give two different diagrams, e.g. (4.26) and (4.27). However the mathematical expressions corresponding to these two diagrams are the same. Thus the Fadeev-Popv contribution also needs to be multiplied by 2, hence the result (6.40) needs to be multiplied by 2. This doesn't affect the correspondence with the "covariant background field method", except that the extra factor of 2 due to the two fundamental representation "sides" of an adjoint representation phase factor, now occurs in (6.40) as well. Hence the counterterm-action for the separate set of SU(N) gauge fields and Fadeev-Popov fields we introduce for each window, is the same as the standard SU(N) BPHZ counterterm action, as we should of course expect.)

We now define, for each such vacuum bubble b drawn on the 2-sphere, X_b to be the result of doing the path integrals over all the 45-paths of b, with a window weight $e^{-\mu^2 B_i}$

for each window i of b, (where B_i is the area of the absolute minimal-area orientable spanning surface of the boundary of window i of b), subject to the mean position of all the vertics in all the 45-paths of b, having the fixed value z. Of course, by translation invariance, X_b is independent of z. X_b *includes* any symmetry factor for b, (for example, for the vector-boson one-loop vacuum bubble, X_b includes the symmetry factor $\frac{1}{2}$.) X_b also *includes* the factor $\frac{\mathrm{d}}{\mathrm{d}M}\mathbf{C}(M)\Big|_{M=1}$, where $\mathbf{C}(M)$, the chromatic polynomial of the bubble b, is by definition the number of distinct ways of colouring the windows of b with M colours available, subject to the constraint that if two windows share a common "boundary", (i.e. a common 45-path), they are to be coloured in two *different* colours, and also to the special rules, discussed in connection with (3.23) - (3.27), for colouring windows that meet at a quartic vertex that has two Fadeev-Popov "legs". Now this $\mathbf{C}(M)$ is completely unaffected by making holes in the windows of b, hence this factor $\frac{\mathrm{d}}{\mathrm{d}M}\mathbf{C}(M)\Big|_{M=1}$ is identical to the corresponding factor $\frac{\mathrm{d}}{\mathrm{d}M}\mathbf{C}(M)\Big|_{M=1}$ for *every* island diagram obtained from b by making holes in its windows.

We also define n_b to be the number of windows of b.

We thus find immediately that, within the approximation that we can set $z_1 = z_2 = z$, the contribution of all the Type-1 island diagrams, when we substitute our ansatz into the right-hand side of the group-variation equation for $f_0(W_{1L}, \ldots, W_{nL}, g^2)$, is equal to

$$f^2 \left(\sum_b n_b X_b \right) f_0(W_{1L}, \ldots, W_{nL}, g^2)$$

times the sum, over all the straight-line segments in the minimal-length spanning tree T_L of $S_{1L}, \ldots, S_{nL}$, of the integral over z of (7.3), where x and y are interpreted as the ends of the straight-line segment of T_L concerned, divided by the factor $\sqrt{\frac{m}{32\pi^3|x-y|^3}}e^{-m|x-y|}$ that our ansatz for $f_0(W_{1L}, \ldots, W_{nL}, g^2)$ includes for the segment $\overline{xy}$. Now the integral over z of (7.3) is dominated by the zone where z is close to the straight line $\overline{xy}$. We choose a coordinate system with origin at x, and one axis pointing from x to y, and define u to be the component of z in this direction, and v to be the three-vector comprising the other three components of z. Then in the exponent we may expand:

$$m\,|x-z| + m\,|y-z| = m\,|x-y| + \frac{|x-y|\,v^2}{2u(|x-y|-u)} + \text{ order } (|v|^4) \qquad (7.4)$$

while in the pre-exponential factors we may approximate $|x-z|$ as u and $|y-z|$ as $(|x-y|-u)$. The Gaussian integral over v then cancels all u-dependence in the pre-

exponential factors, so the u-integral then simply gives the factor $|x-y|$. Thus for the z-integral of (7.3) we get:

$$\frac{|x-y|}{2m}\sqrt{\frac{m}{32\pi^3\,|x-y|^3}}e^{-m|x-y|} \tag{7.5}$$

which is simply $\frac{|x-y|}{2m}$ times the factor that our ansatz for $f_0(W_{1L},\ldots,W_{nL},g^2)$ includes for the segment $\overline{xy}$. (As a check, we note that, if we were calculating the z-integral of (7.3), times a mass counterterm $-2m(\delta m)$, then by (7.5) we would find correctly, that for large $|x-y|$, and to first order in δm, we get the correction to the large $|x-y|$ asymptotic form (5.2) of the free massive scalar propagator, for the mass shift $m \to m+(\delta m)$.)

We thus see that, remembering that we defined the lengths of the separate straight lines forming the minimal-length spanning tree T_L of $S_{1L},\ldots,S_{nL}$ to be $b_{1L},\ldots,b_{qL}$, where $(n-1) \leq q \leq (2n-3)$, and within the approximation that we can set $z_1 = z_2 = z$, the contribution of all the Type-1 island diagrams, when we substitute our ansatz into the right-hand side of the group-variation equation for $f_0(W_{1L},\ldots,W_{nL},g^2)$, and multiply both sides of the group variation equation by $-\frac{2\beta(g)}{g}$, is:

$$-\frac{2\beta g}{g}\frac{(b_1+\ldots+b_q)\,L}{2m}f^2\left(\sum_b n_b X_b\right)f_0(W_{1L},\ldots,W_{nL},g^2) \tag{7.6}$$

which has exactly the same functional form as (7.2).

Now what about the contributions of the other island diagrams? We now *change* our terminology for non-Type-1 island diagrams. Whereas before, we called *all* non-Type-1 island diagrams, "Type-2", we now define, for all integers $c \geq 1$, a "Type-c island diagram" to be, precisely, an island diagram obtained from one of our connected, one-line irreducible vacuum bubbles, formed of 45-paths, and drawn on the 2-sphere with no lines crossing, as above, by making one or more holes in each of c different windows of the vacuum bubble. Thus Type-1 island diagrams are as before, but in general, for each $c \geq 1$, a Type-c island diagram is an island diagram where precisely c of the windows of the diagram are *not* simply connected. (We recall that, as discussed after examples (4.1) - (4.73), no contributing island diagram, i.e. no island diagram for which $\frac{\mathrm{d}}{\mathrm{d}M}\mathbf{C}(M)\Big|_{M=1}$ is nonzero, has any 45-paths that do *not* form part of the island.) The examples of *contributing* Type-2 island diagrams are now (4.55), (4.71), and (4.72), and the simplest example of a Type-3 island diagram is now (4.73). We note that a Type-c island diagram requires making at least c holes in the windows of the vacuum

bubble, hence a Type-c island diagram can only occur for $f_0(W_{1L}, \ldots, W_{nL}, g^2)$ such that $n \geq c$, and furthermore, that a Type-c island diagram of course has at least c windows.

Now of course we are at present considering $f_0(W_{1L}, \ldots, W_{nL}, g^2)$ such that $n \geq 2$. Let us first consider the contributions of Type-2 island diagrams to the right-hand side of the group-variation equation for $f_0(W_{1L}, \ldots, W_{nL}, g^2)$. Now a Type-2 island diagram has holes in precisely two of the windows of the corresponding vacuum bubble, and thus partitions the W_{iL}'s into two nonempty sets, corresponding to which of the W_{iL}'s occur in which of the two relevant windows of the vacuum bubble. Now a partition of the external states of a process into two nonempty sets defines a channel for that process, and if that process is described by a particular tree diagram, (or, in our case, our minimal-length spanning tree T_L), each given channel may or may not correspond to a propagator of that tree diagram. Now, with the same assumptions as we made in studying the contributions of the Type-1 islands, namely, that m is strictly smaller than the mass M of the lowest-mass state of the cylinder with two 45-paths along it, and furthermore, (for simplicity), that m is sufficiently smaller than M that there is no significant tendency for islands to elongate along segments of T_L, we readily see that, since all the individual straight-line segments of T_L are, by assumption, much longer than the predominant island size $\frac{1}{\mu}$, if the partition of $\{W_{1L}, \ldots, W_{nL}\}$ defined by a given Type-2 island diagram corresponds to one of the straight-line segments of T_L, (in the sense that cutting that segment of T_L partitions $\{W_{1L}, \ldots, W_{nL}\}$ into the same two nonempty parts as defined by that Type-2 island diagram), then that Type-2 island diagram *does* give a significant contribution, namely from configurations where the island "slides up and down" that particular straight line segment of T_L, while if the partition of $\{W_{1L}, \ldots, W_{nL}\}$ defined by that Type-2 island diagram does *not* correspond to any of the straight-line segments of T_L in that sense, then that Type-2 island diagram does *not* give a significant contribution, because no matter what configuration the island is in, either the island is stretched to a size large compared to $\frac{1}{\mu}$, or else the relevant minimal-length spanning trees have a total length (of all their segments) significantly greater than the total length of T_L. Thus we can associate each Type-2 island diagram that gives a significant contribution, with a particular segment of T_L, namely the segment of T_L that defines the same partition of $\{W_{1L}, \ldots, W_{nL}\}$ as that Type-2 island diagram.

Now, as we know, each Type-2 island diagram is obtained from a corresponding

vacuum bubble by making the appropriate holes in the appropriate two windows of that vacuum bubble. We readily see, in a manner precisely analogous to what we found for the Type-1 islands, that if a Type-2 island diagram defines a partition of $\{W_{1L}, \ldots, W_{nL}\}$ that matches that defined by a segment of T_L of length $b_j L$, then after multiplying by $-\frac{2\beta(g)}{g}$, its contribution to the right-hand side of the group-variation equation for $f_0(W_{1L}, \ldots, W_{nL}, g^2)$ is:

$$-\frac{2\beta(g)}{g}\frac{b_j L}{2m} f^2 X_b f_0(W_{1L}, \ldots, WnL, g^2) \tag{7.7}$$

where the definition of X_b is exactly as before. We note, however, that if, due to symmetries of the vacuum bubble b, there are several distinct ways of obtaining the given Type-2 island diagram by putting the appropriate holes in two distinct windows of b, then (7.7) just gives the contribution of *one* of those ways to the contribution of that Type-2 island diagram. Now for a given straight-line segment of T_L, (and the corresponding partition of $\{W_{1L}, \ldots, W_{nL}\}$ into two nonempty parts), and a given one of our vacuum bubbles b, there is a total of $n_b(n_b - 1)$ such contributions (7.7) to the right-hand side of the group-variation equation for $f_0(W_{1L}, \ldots, W_{nL}, g^2)$, corresponding to the $n_b(n_b - 1)$ different ways of assigning the two parts into which $\{W_{1L}, \ldots, W_{nL}\}$ is partitioned, to two distinct windows of b, (where n_b denotes, as before, the number of windowsw of b). We note that (7.7) has no dependence on the particular segment of T_L concerned, other than through the length $b_j L$ of that segment. In particular, (7.7) does *not* depend on the particular partition of $\{W_{1L}, \ldots, W_{nL}\}$ defined by that segment. Thus we see that the total effect of adding, to the contributions (7.6) of all the Type-1 island diagrams, the contributions of all the Type-2 island diagrams, is simply to replace n_b by n_b^2 in (7.6). In other words, the total contribution of all the Type-1 island diagrams and all the Type-2 island diagrams, to the right-hand side of the group-variation equation for $f_0(W_{1L}, \ldots, W_{nL}, g^2)$, is:

$$-\frac{2\beta(g)}{g}\frac{(b_1 + \ldots + b_q)\, L}{2m} f^2 \left(\sum_b n_b^2 X_b\right) f_0(W_{1L}, \ldots, W_{nL}, g^2) \tag{7.8}$$

Furthermore, we readily see that *no* island diagram of Type-3, or any higher type, makes any significant contribution to the right-hand side of the group-variation equation for $f_0(W_{1L}, \ldots, W_{nL}, g^2)$, for no matter what configuration the isand is in, either the island is stretched to a size large compared to $\frac{1}{\mu}$, or the minimal-length spanning trees concerned have a total length significantly larger than the total length of T_L,

(i.e. the total length of the segments of T_L), or else the island is "anchored" at one of the vertices of T_L where three straight segments of T_L meet. (This third possibility arises for a Type-3 island diagram when the partition of $\{W_{1L}, \ldots, W_{nL}\}$ into three nonempty parts, as defined by that Type-3 island diagram, matches the partition of $\{W_{1L}, \ldots, W_{nL}\}$ into three nonempty parts defined by removing that vertex from T_L. In this case, the island can be small, and the total length of the spanning trees involved be no longer than the total length of T_L, provided the island is close to that vertex. However, since, as shown in connection with (4.94) - (4.98) above, the angles between the three segments meeting at that vertex will all be $\frac{2\pi}{3}$, it is impossible for the island to move away from that vertex without either stretching to a size significantly greater than $\frac{1}{\mu}$, or the total length of the spanning trees involved becoming significantly greater than the total length of T_L. Thus these cases cannot produce the extra factor of L that occurs in (7.2) and (7.8).)[1]

Thus, since the non-island diagrams also do not make any significant contributions to the right-hand side of the group-variation equation for $f_0(W_{1L}, \ldots, W_{nL}, g^2)$ in the region we are investigating, (i.e. no contributions with the extra power of L that occurs in (7.2) and (7.8)), we wee that, subject to ur assumption that the mass m of the lightest glueball is strictly less than the mass M of the lightest state of the cylinder with two 45-paths along it, and our further assumption, made for simplicity, that m is sufficiently smaller than M that there is no significant tendency for islands to elongate along the straight-line segments of T_L, the result of substituting our ansatz into the right-hand side of the group-variation equation for $f_0(W_{1L}, \ldots, W_{nL}, g^2)$, when the separations of $W_{1L}, \ldots, W_{nL}$ are all large compared to $\frac{1}{\mu}$, and m times the total length of all the straight-line segments in T_L is large compared to μ^2 times the sum of the areas of the absolute minimal-area spanning surfaces $S_{1L}, \ldots, S_{nL}$, of $W_{1L}, \ldots, W_{nL}$, is given by (7.8). And comparing (7.8) with (7.2), we see that our ansatz does indeed give a solution of the group-variation equations in this domain, and that, furthermore, the mass m of the lightest glueball is given by:

$$m^2 = \frac{\beta(g)}{g} f^2 \left(\sum_b n_b^2 X_b \right) \tag{7.9}$$

Here g^2 is of course to be set equal to the critical value that it takes throughout the domain where our ansatz applies, as determined by (5.10), or more precisely, by the

[1]Note added: example (4.73) is a simple case in which this situation occurs.

corrected version of (5.10) to be given below, and as discussed after (5.37). And of course, since $\frac{\beta(g)}{g}$ is negative, $\sum_b n_b^2 X_b$ must be negative, and, indeed, there is no doubt that the sum X_b for the vector-boson and Fadeev-Popov one-loop vacuum bubbles, (which have $n_b = 2$), must be negative.

We next note that if, instead of assuming that the mass m of the lightest glueball is strictly less than the mass M of the lowest-mass state of the cylinder with two 45-paths along it, we had assumed that m is *equal* to M, we would have found that our ansatz did *not* solve the group-variation equations. This can be seen in the example of the group-variation equation for $f_0(W_{1L}, W_{2L}, g^2)$, in the domain where the separation, approximately L, between W_{1L} and W_{2L} is large compared to $\frac{1}{\mu}$, and mL is large compared to μ^2 times the sum of the areas of the absolute minimal-area spanning surfaces of W_{1L} and W_{2L}. In this domain the assumption that $m = M$ means that there is no exponential suppression of the island size along the straight line between W_{1L} and W_{2L}: we have to consider configurations of the one-loop islands (4.53) and (4.55) where the islands are "stretched" to lengths of order L along the straight line between W_{1L} and W_{2L}. And for the higher-loop islands we have to consider configurations where the vertices of the island are divided into two nonempty groups, with the two groups being connected to one another by just two 45-paths, and stretching the island such that one of these two groups moves towards W_{1L} and the other moves towards W_{2L}. And we also have to consider possible "stretchings" of the higher-loop islands in two or more "two-path connected" regions. But such "stretched" islands produce *too many* extra powers of L in the right-hand side of the group-variation equation for $f_0(W_{1L}, W_{2L}, g^2)$ to match the single extra power of L in the left-hand side, as given by (7.2). Indeed, considering just the one-loop islands (4.53) and (4.55), (it doesn't matter which), if we consider summing over all 45-paths of a stretched island configuration such that the two "ends" of that stretched island are fixed at z_1 and z_2, where z_1 and z_2 are close to the straight line between W_{1L} and W_{2L}, but may be separated from one another by a distance of order L, then the assumption that $M = m$ implies that the result of summing over those 45-paths, subject to z_1 and z_2 fixed, is a constant numerical multiple of $\sqrt{\frac{m}{32\pi^3|z_1-z_2|^3}}e^{-m|z_1-z_2|}$. And if z_1 is the end of the island nearer W_{1L}, z_2 is the end of the island nearer W_{2L}, and x and y are the ends, on the absolute minimal-area orientable spanning surfaces of W_{1L} and W_{2L} respectively, of the shortest straight-line segments between those two spanning surfaces, then our ansatz gives a factor $\sqrt{\frac{m}{32\pi^3|x-z_1|^3}}e^{-m|x-z_1|}$ for the domain between W_{1L} and z_1, and a

factor $\sqrt{\frac{m}{32\pi^3|z_2-y|^3}}e^{-m|z_2-y|}$ for the domain between z_2 and W_{2L}. We then find, doing the z_1 and z_2 integrals in exactly the same way as before, (i.e. to arrive at (7.5)), we get *two* extra powers of L, (essentially one extra power of L for each end of the island). And when we consider higher-loop island diagrams with several "stretching" regions, we get even more extra powers of L. This does *not* match (7.2), and there is certainly no reason to expect that all the trms with the extra powers of L would exactly cancel one another. We conclude that the mass m of the lightest glueball must be *strictly* less than the mass M of the lowest-mass state of the cylinder with two 45-paths along it.

7.2 The area-law domain

Having now seen that our ansatz does indeed satisfy the group-variation equations in the "glueball" domain of widely separated loops, we return to the consideration of the "area law" domain. We considered this domain qualitatively before, obtaining equation (5.10), but, as we mentioned in the paragraph after (5.10), that qualitative estimate was in fact not quite right. We consider a loop W_1, such that the largest distance between any two points of W_1, (i.e. the "diameter" of W_1), is R, where R is the onset point of the smooth long-distance cutoffs we impose on the propagators in our BPHZ counterterms, and define W_{1L} to be the loop obtained from W_1 by scaling W_1 by the factor $\frac{L}{R}$. Thus W_{1L} is identical in shape to W_1, but differs in size, having the diameter L. We define the area of the absolute minimal-area orientable spanning surface of W_1 to be a_1R^2, so that the area of the absolute minimal-area orientable spanning surface of W_{1L} is a_1L^2. we consider the group-variation equation for $f_0(W_{1L}, g^2)$, and note that, exactly as before, (equation (5.8)), when we substitute our ansatz into the left-hand side of the group-variation equation for $f_0(W_{1L}, g^2)$, and multiply by $-\frac{2\beta(g)}{g}$, we get $-2\mu^2 a_1 L^2 f_0(W_{1L}, g^2)$. Furthermore, exactly as before, (in the discussion before (5.5)), we may expect, for large L, the contribution of each *island* diagram to the right-hand side of the group-variation equation for $f_0(W_{1L}, g^2)$, to behave *roughly* as $L^2 f_0(W_{1L}, g^2)$, while the contribution of each *non*-island diagram will behave only as $Lf_0(W_{1L}, g^2)$. Thus, for large L, we only need to consider the contributions of the *island* diagrams. (Of course, only *Type-1* island diagrams occur in the right-hand side of the group-variation equation for $f_0(W_{1L}, g^2)$.)

We consider an island diagram contributing to the right-hand side of the group-variation equation for $f_0(W_{1L}, g^2)$, and define W_2 to be the outer boundary of the island.

Then the window weight for the non-simply connected window that "surrounds" the island is given by $f_0(W_{1L}, W_2, g^2)$. Now due to the effective mass 1.3μ of the 45-paths, the predominant contributions for this island diagram will be given, for large L, by islands of size $\frac{1}{\mu}$, hence the predominant contributions will come from configurations where the diameter of W_2 is roughly $\frac{1}{\mu}$. Now according to our ansatz, as stated after (5.3), the functional form of $f_0(W_{1L}, W_2, g^2)$ depends on whether the absolute minimal-area orientable spanning surface S of W_{1L} and W_2 has cylinder topology, (i.e. a sphere with two holes in), or consists of the separate absolute minimal-area orientable spanning surfaces S_{1L} and S_2 of W_{1L} and W_2. Now clearly the configurations of W_2 which give the largest values of $f_0(W_{1L}, W_2, g^2)$, are those where W_2 is a simple loop lying within the minimal-area orientable spanning surface S_{1L} of W_{1L}, and oriented consistently with the orientation of S_{1L} defined by W_{1L}, (i.e. such that if W_{1L} were shrunk until it coincided with W_2, then the arrows on the two loops would point in *opposite* directions), for the area A of S is then equal to $A_{1L} - A_2$, where A_{1L} is the area of S_{1L}, and A_2 is the area of S_2. Thus for a first quantitative estimate we shall assume that the predominant contributions come from W_2 such that S has cylinder topology, and then check the validity of this assumption after the calculation.

To examine a specific example, we choose 4-dimensional Cartesian coordinates (w, x, y, z), and assume that W_{1L} is the circle of diameter L given parametrically by $\frac{L}{2}(\cos\theta, \sin\theta, 0, 0)$, $0 \leq \theta \leq 2\pi$, and W_2 is the circle of diameter $\frac{1}{\mu}$ given parametrically by $(w, x, y, z) + \frac{1}{2\mu}(\cos\theta, -\sin\theta, 0, 0)$, $0 \leq \theta \leq 2\pi$. We assume that $\sqrt{w^2 + x^2} + \frac{1}{2\mu}$ is less than $\frac{L}{2}$, and that y and z are sufficiently small that the area A of the absolute minimal-area orientable spanning surface S of W_{1L} and W_2 is given to a good approximation by the appropriate solution of Laplace's equation in the circle bordered by W_{1L}. (We will also check the validity of this assumption afterwards.) We then find immediately, from the appropriate solution of Laplace's equation, that in the limit of large μL, and without any restriction on w and x, other than that given above, the area A of S is given by:

$$A = \frac{\pi}{4}\left(L^2 - \frac{1}{\mu^2}\right) + \frac{\pi\,(y^2 + z^2)}{\ln\left(\mu L\left(1 - \frac{4(w^2+x^2)}{L^2}\right)\right)} \tag{7.10}$$

Now, with W_{1L} still being our circle of diameter L, but W_2 now unrestricted, we define $J(w, x, y, z)$ to be the contribution to our island diagram from all configurations of the island such that the mean position of all the vertices of the 45-paths of the island is equal to (w, x, y, z). Thus the contribution from this island diagram to the right-hand

side of the group-variation equation for $f_0(W_{1L}, g^2)$ is $\int\int\int\int \mathrm{d}w\mathrm{d}x\mathrm{d}y\mathrm{d}z J(w,x,y,z)$. Now of course $J(w,x,y,z)$ is largest when (w,x,y,z) lies in, or close to, S_{1L}, i.e. the circular disc of diameter L that fills W_{1L}. Let us suppose that (w,x,y,z) lies in, or close to, S_{1L}. Then $J(w,x,y,z)$ will certainly get contributions from island whose outer boundary W_2 is such that the absolute minimal-area orientable spanning surface S of W_{1L} and W_2 does *not* have cylinder topology, but consists instead of S_{1L} and the absolute minimal-area orientable spanning surface S_2 of W_2. For example, W_2 could be the same circle we considered before, but with its arrow pointing in the other direction, i.e. with its parameter θ replaced by $-\theta$. However the contributions of such island configurations will be suppressed, due to S having larger area, relative to the contributions of island configurations such that the projection onto S_{1L} of the outer boundary W_2 of the island, forms a simple loop in S_{1L}, (i.e. a loop without any self-intersections), oriented consistently with S_{1L}, and such that, for *all* points on W_2, the perpendicular distance from that point to S_{1L} is small. In fact, the predominant contributions to $J(w,x,y,z)$ will come from island configurations such that the outer boundary W_2 of the island has the properties just described, and furthermore the projections onto S_{1L} of all the remaining 45-paths of the island, (if the island has two or more loops), all lie within the simple loop formed by the projection of W_2 onto S_{1L}, and intersect neither one another nor themselves, and furthermore, for *every* point on *every* 45-path of the island, the perpendicular distance from that point to S_{1L} is small. Now for such island configurations, the absolute minimal-area orientable spanning surfaces of *every* window of the diagram, are given to a good approximation by the appropriate solutions of Laplace's equation, in the zones of S_{1L} delineated by the projections onto S_{1L} of the 45-paths of the island. Thus the total of the areas of the absolute minimal-area orientable spanning surfaces of the windows of the diagram, is equal to $\frac{\pi L^2}{4}$, plus corrections quadratic in the perpendicular distances of the points of the 45-paths of the island from S_{1L}. Furthermore, due to the effective mass 1.3μ of the 45-paths, the predominant contributions come from islands of size roughly $\frac{1}{\mu}$, and by assumption, L is large compared to $\frac{1}{\mu}$. Furthermore, due to the fact that we are seeking the contribution of this island diagram to the right-hand side of the group-variation equation for $f_0(W_{1L}, g^2)$ at large L, and this contribution behaves *roughly* as $L^2 f_0(W_{1L}, g^2)$ at large L, we may exclude the contributions of islands whose projection onto S_{1L} lies within a band of fixed width B at the edge of S_{1L}, where B is large compared to $\frac{1}{\mu}$, but independent of L as L becomes large, since the contribution of the

excluded island configurations behaves only as $Lf_0(W_{1L}, g^2)$ as L becomes large. Then, due to the facts that our islands are small, and their projections onto S_{1L} are not too close to the edge of S_{1L}, the only dependence, on the coordinates w and x of the mean position (w, x, y, z) of the vertices of the 45-paths of the island, of the coefficients of the terms in the window areas quadratic in the perpendicular distances of the points of the 45-paths of the island from S_{1L}, is in the coefficients of the terms quadratic in y and z in the area of the absolute minimal-area orientable spanning surface S of the non-simply connected window, bordered by W_{1L} and W_2, that "surrounds" the island. In fact, the *only* dependence on w, x, y, and z, of the window areas, is to a good approximation given by the terms quadratic in y and z in the area (7.10) of A. Indeed, the areas of the minimal-area spanning surfaces of the *internal* windows of the island, depend only on the 45-paths bordering the windows concerned, and are completely independent of w, x, y, and z. For the non-simply connected window, we note that Laplace's equation is linear, and that if we choose polar coordinates (r, θ) in the plane of W_{1L}, with origin at some point *strictly* in the interior of the projection of the island onto S_{1L}, (so the origin of the polar coordinates will *not* in general coincide with the origin of the Cartesian coordinates (w, x)), then the general solution of Laplace's equation in the zone of S_{1L} outside the projection of the island onto S_{1L}, is given by:

$$a + b \ln r + \sum_{n=1}^{\infty} c_n r^n \cos(n(\theta - \gamma_n)) + \sum_{n=1}^{\infty} d_n r^{-n} \cos(n(\theta - \delta_n)) \tag{7.11}$$

where a, b, the c_n, $(n \geq 1)$, and the d_n, $(n \geq 1)$, are constant coefficients, and the γ_n and the δ_n, $(n \geq 1)$, are constant angles, chosen to satisfy the boundary conditions. (Each of the constant coefficients is actually a 2-vector corresponding to the $4-2 = 2$ dimensions perpendicular to S_{1L}, and each of the constant angles also has an undisplayed index indicating which of the two dimensions perpendicular to S_{1L} it refers to, but we can treat each of the two dimensions perpendicular to S_{1L} independently of the other one.) We define the $\theta = 0$ direction for our polar coordinates to be the direction opposite to the direction given by an arrow pointing from the origin of our polar coordinates to the centre of W_{1L}. We assume that the origin of our polar coordinates is at the projection onto S_{1L} of the mean position of the vertices of the island, and thus has the cartesian coordinates (w, x), and we define $s \equiv \sqrt{w^2 + x^2}$. Then if, for example, W_2 is the circle specified before (7.10), then in the limit of large μL, and with no restrictions on w and x other than that $s + \frac{1}{2\mu}$ be less than $\frac{L}{2}$, which ensures that the projection of W_2 onto the plane of W_{1L} lies within S_{1L}, we find that, (remembering that the coefficients are

2-vectors, and that the constant angles also have an undisplayed index):

$$b = \frac{-(y,z)}{\ln\left(\mu L\left(1-\frac{4s^2}{L^2}\right)\right)}, \qquad a = -b\ln\left(\frac{L}{2}\left(1-\frac{4s^2}{L^2}\right)\right)$$

$$c_n = \frac{b}{n}\left(\frac{4s}{L^2-4s^2}\right)^n,\ (n\geq 1), \qquad d_n = 0,\ (n\geq 1)$$

$$\gamma_n = \delta_n = 0,\ (n\geq 1) \tag{7.12}$$

and (7.11) then becomes:

$$b\left(\ln\left(\frac{2r}{L}\right) - \frac{1}{2}\ln\left(\left(1-\frac{4s^2}{L^2}\right)^2 + \left(\frac{4rs}{L^2}\right)^2 - \frac{8rs}{L^2}\left(1-\frac{4s^2}{L^2}\right)\cos\theta\right)\right) \tag{7.13}$$

We readily verify that (7.13) vanishes exactly on W_{1L}, i.e. when $\sqrt{r^2+s^2+2rs\cos\theta} = \frac{L}{2}$, or in other words, when $\left(1-\frac{4s^2}{L^2}\right) = \frac{4r^2+8rs\cos\theta}{L^2}$, and on the circle $r = \frac{1}{2\mu}$, becomes equal to $(y,z)\left(1+\ \text{order}\left(\frac{1}{\mu L}\right)\right)$.

Now with ∂_μ representing the 2-vector $\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial x}\right)$, and denoting (7.11) by $(\tilde{y},\tilde{z})$, the contributions to the area A of the absolute minimal-area orientable spanning surface S of the non-simply connected window, from the components of the island configuration perpendicular to the plane of S_{1L}, are given, through quadratic order in those components, by the Laplace action:

$$\frac{1}{2}\int\int \mathrm{d}w\mathrm{d}x\left(\left(\partial_\mu\tilde{y}\right)\left(\partial_\mu\tilde{y}\right) + \left(\partial_\mu\tilde{z}\right)\left(\partial_\mu\tilde{z}\right)\right) = \frac{1}{2}\int \mathrm{d}t\left(\tilde{y}\left(n.\partial\tilde{y}\right) + \tilde{z}\left(n.\partial\tilde{z}\right)\right) \tag{7.14}$$

where $\mathrm{d}t$ represents the boundary element of the projection onto S_{1L} of the non-simply connected window, n denotes the outward unit normal, in the plane of S_{1L}, at each point of the boundary of the projection onto S_{1L} of the non-simply connected window, and the integral in the right-hand side of (7.14) goes over both the outer boundary, (i.e. W_{1L}), of the projection onto S_{1L} of the non-simply connected window, and the inner boundary of that domain, (i.e. the projection onto S_{1L} of W_2), and Laplace's equation was of course used in obtaining the right-hand side of (7.14).

We readily verify that substituting (7.12) and (7.13) into (7.14), gives the terms in (7.10) quadratic in y and z, plus terms of order $\frac{\left(y^2+z^2\right)}{\mu L}$. (Only the inner boundary contributes, since (7.13) vanishes exactly on the outer boundary.)

Now for more general island configurations, subject to the requirement that Laplace's equation be approximately applicable to the calculation of the "out-of-plane" contributions to the area, (which, as we noted above, will be true for the most important island

configurations), the "out-of-plane" components $\tilde{y}$ and $\tilde{z}$ of the absolute minimal-area orientable spanning surface S of the non-simply connected windows, will still be given by (7.11), with a, b, and the c_n, $(n \geq 1)$, as given by (7.12), but with the differences that the 2-vector coefficients d_n, $(n \geq 1)$, will in general now be non-zero and of order $\mu^{-(n+1)}$, (since both the "in-plane components" and the "out-of-plane components" of the *internal* coordinates of the island are of size roughly $\frac{1}{\mu}$), the angles γ_n and δ_n, $(n \geq 1)$, may be non-zero, and that the w and x that occur in (7.12) via $s = \sqrt{w^2 + x^2}$, and the y and z that occur in (7.12), may differ from the components (w, x, y, z) of the mean position of the vertices of the island, by amounts of order $\frac{1}{\mu}$. Now we readily verify that, if the projection of the outer boundary W_2 of the island onto S_{1L} is circular, and (w, x) are the "in-plane" components of the centre of this circle, and moreover y and z are equal to the mean "out-of-plane" components of W_2, calculated with equal weights assigned to equal elements of the circular projection of W_2 onto S_{1L}, then when we substitute (7.11) into (7.14), all cross terms between a, b, and the c_n's on the one hand, (which are proportional to y and z), and the d_n's on the other hand, (which are proportional to the "out-of-plane" components of the *internal* coordinates of the island), are suppressed by at least one factor of $\frac{1}{\mu L}$, which means that for large μL the entire dependence of the area A on y and z is given by the terms in (7.10) quadratic in y and z, as modified by multiplying μ inside the logarithm by a factor of order 1, to get the correct radius of the circle. Moreover, since the coefficients d_n do not depend on w and x, and they are all suppressed by at least on factor of $\frac{1}{\mu L}$ on the outer boundary W_{1L}, the entire dependence of the area A on w and x is also given, for large μL, by the terms in (7.10) quadratic in y and z, (with other terms dependent on w and x all being suppressed by at least one factor of $\frac{1}{\mu L}$). Now of course the projection of the outer boundary W_2 of the island onto S_{1L} will never be exactly circular in practice, and the weighting of the vertices of the 45-paths of the island in the calculation of y and z is different from the weighting just described, but nevertheless there will in general be, within the convex hull of the 45-paths of the island, a point (w, x, y, z), such that, for large μL, the *only* dependence of the area A on w, x, y, and z, is through the term in (7.10) quadratic in y and z, with μ inside the logarithm being multiplied by a factor of order 1, (with all other terms dependent on any of w, x, y, and z, all being suppressed by at least one factor of $\frac{1}{\mu L}$). Here (w, x), and the factor of order 1 by which we multiply μ inside the logarithm, are determined respectively by the centre and the diameter of a circle, roughly coincident with the projection onto S_{1L} of the island, and

roughly of diameter $\frac{1}{\mu}$, onto which we transform the projection onto S_{1L} of the outer boundary of the island, by a suitable conformal transformation, leaving W_{1L} invariant, of the projection onto S_{1L} of the non-simply connected window. (We could, of course, by another conformal transformation, move the circle representing the island to the centre of S_{1L}, but the radius of that circle then gets divided by precisely the w and x dependent factor in the logarithm in (7.10), so that we get the same result again.) And y and z are determined by the mean values of the appropriate "out-of-plane" components of the points of W_2, with the weights in the mean being determined by the above conformal transformation, such that the elements of W_2 whose projections onto S_{1L} are transformed to equal-sized elements of the circle by the conformal transformation, get equal net weights.

Thus we see that, for large μL, the entire dependence of $J(w, x, y, z)$, (which, as defined after (7.10), is the contribution to our island diagram from all configurations of the island such that the mean position of the vertices of the 45-paths of the island is equal to (w, x, y, z)), on w, x, y, and z, is to a good approximation given by a factor equal to the exponential of $-\mu^2$ times the terms in (7.10) quadratic in y and z, with corrections being suppressed by at least one factor of $\frac{1}{\mu L}$. In fact, for large μL, $J(w, x, y, z)$ is to a good approximation equal to $\tilde{J} \equiv J(0, 0, 0, 0)$, times the exponential of $-\mu^2$ times the terms in (7.10) quadratic in y and z, with the corrections all being suppressed by at least one power of $\frac{1}{\mu L}$. Furthermore, the entire dependence of $\tilde{J}$ on L is through the factor $e^{-\mu^2 \frac{\pi}{4} L^2}$, or in other words, the factor $f_0(W_{1L}, g^2)$, which also arises from the exponentiation of $-\mu^2$ times (7.10). Indeed, for large μL, the sum over the 45-paths of the island, subject to their mean position (w, x, y, z) being held fixed, is to a good approximation completely independent of L: the weights associated with the internal windows of the island are manifestly independent of L, while if w, x, y, and z are defined as described above, to avoid any cross-terms, at leading order in $\frac{1}{\mu L}$, between a, b, and the c_n's on the one hand, and the d_n's on the other hand, occurring in A, then the only effect of the non-simply connected window on the sums over the 45-paths of the island, subject to (w, x, y, z) being held fixed, is through the coefficients d_n, which are independent of L at leading order in $\frac{1}{\mu L}$, and the angles γ_n and δ_n. And if w, x, y, and z are defined simply as the coordinates of the mean positon of the vertices of the island, they will differ by amounts of no more than $\frac{1}{\mu}$ from the values they would have to take to avoid the cross terms. Hence at leading order in $\frac{1}{\mu L}$, all dependence on L of the sum over the 45-paths of the island, subject to their mean position (w, x, y, z)

being held fixed, may be absorbed by adjustments of w, x, y, and z, by amounts of at most $\frac{1}{\mu}$.

Thus if we define K by:

$$\tilde{J} = ke^{-\mu^2 \frac{\pi}{4} L^2} = K f_0(W_{1L}, g^2) \tag{7.15}$$

at one particular value of L, large compared to $\frac{1}{\mu}$, so that K is completely independent of L, and of w, x, y, and z, then at leading order in $\frac{1}{\mu L}$, the entire dependence of $J(w,x,y,z)$, on L, and on w, x, y, and z, is to a good approximation expressed by the formula:

$$J(w,x,y,z) = K\ f_0(W_{1L}, g^2)\ \exp\left(\frac{-\mu^2 \pi\,(y^2+z^2)}{\ln\left(v\mu L\left(1-\frac{4(w^2+x^2)}{L^2}\right)\right)}\right) \tag{7.16}$$

Here v is an absolutely fixed numerical constant of order 1, that represents an effective mean value of the factor of order 1, by which we multiply μ inside the logarithm, as determined, for each important configuration of the outer boundary W_2 of the island, by the diameter of the circle, roughly of diameter $\frac{1}{\mu}$, and roughly coincident with the projection onto S_{1L} of the island, onto which we transform the projection onto S_{1L} of the outer boundary of the island, by a conformal transformation as above. We note that (7.16) is valid to a good approximation due to the fact that, at leading order in $\frac{1}{\mu L}$, the actual slight additional dependencies of $J(w,x,y,z)$ on L, w, x, y, and z, not included in (7.16), may all be reproduced by small adjustments of w, x, y, and z in (7.16), *by amounts of at most* $\frac{1}{\mu}$, with the actual values of these small adjustments of w, x, y, and z in (7.16), themselves being dependent on L, w, x, y, and z.

We can now immediately determine, from (7.16), the dependence on L, at large μL, of the contribution of our island diagram to the right-hand side of the group-variation equation for $f_0(W_{1L}, g^2)$. Indeed, by the definition, (after (7.10)), of $J(w,x,y,z)$, the contribution of this island diagram to the right-hand side of the group-variation equation for $f_0(W_{1L}, g^2)$, is simply the integral, over (w,x,y,z), of $J(w,x,y,z)$. And from (7.16) we find immediately:

$$\int\int \mathrm{d}y\mathrm{d}z J(w,x,y,z) = \frac{K}{\mu^2} f_0(W_{1L}, g^2) \ln\left(v\mu L\left(1-\frac{4\,(w^2+x^2)}{L^2}\right)\right) \tag{7.17}$$

We note that (7.17) is valid for (w,x) within S_{1L} and not too close to the edge of S_{1L}, (i.e. not too close to W_{1L}), and that, as noted in the discussion before (7.11), we may neglect, at large μL, the contributions of the domains where (7.17) is not valid, since

their contributions will behave only as $Lf_0(W_{1L}, g^2)$ at large L, while the contribution of the domain where (7.17) is valid behaves *roughly* as $L^2 f_0(W_{1L}, g^2)$ at large L.

Finally we have to integrate (7.17) over the domain of (w, x) where it is valid, say over the domain $\sqrt{w^2 + x^2} \leq \frac{L}{2} - B$, where B is a fixed value, large compared to $\frac{1}{\mu}$, but independent of L as L becomes large. We readily find that the integral of (7.17) over this domain is exactly:

$$\frac{\pi K}{\mu^2} f_0(W_{1L}, g^2) \left(\left(\frac{L}{2} - B \right)^2 \ln \left(\frac{v\mu L}{e} \right) - B(L - B) \ln \left(\frac{4B(L - B)}{L^2} \right) \right) \tag{7.18}$$

where of course e denotes the base of natural logarithms. The leading term in (7.18) at large L is:

$$\frac{K}{\mu^2} f_0(W_{1L}, g^2) \frac{\pi L^2}{4} \ln \left(\frac{v\mu L}{e} \right) \tag{7.19}$$

which is of course completely independent of B. Now the preceding calculation applies to *every* island diagram contributing to the right-hand side of the group-variation equation for $f_0(W_{1L}, g^2)$. The only differences between different island diagrams are in the values of the constant K, and the value of the numerical constant v of order 1. Thus (7.19) also gives the leading term at large L in the sum of the contributions of *all* the island diagrams to the right-hand side of the group-variation equation for $f_0(W_{1L}, g^2)$, with K now interpreted as the sum of the K's of the individual island diagrams, and v interpreted as an effective mean value of the v's of the individual island diagrams. Hence, in view of the discussion preceding (7.10), (7.19) also gives the leading term at large L in the sum of *all* the diagrams in the right-hand side of the group-variation equation for $f_0(W_{1L}, g^2)$.

7.3 The need to cancel the extra factor of $\ln(\sigma A)$

We see that (7.19) differs from the result we expected from our qualitative consideration before (5.5), which resulted in (5.10), by the factor $\ln\left(\frac{v\mu L}{e}\right)$. Furthermore, due to this extra factor, (7.19), after multiplication by $-\frac{2\beta(g)}{g}$, (with g^2 of course set equal to the critical value), does *not* match the behaviour $L\frac{\partial}{\partial L} f_0(W_{1L}, g^2) = -2\mu^2 \frac{\pi}{4} L^2 f_0(W_{1L}, g^2)$ of the leading term at large L in the left-hand side of the group-variation equation for $f_0(W_{1L}, g^2)$, after multiplication by $-\frac{2\beta(g)}{g}$.

However, before concluding that our ansatz is wrong, we have to check that our calculation has been correct. We have assumed, firstly, that the absolute minimal-area

orientable spanning surface of the non-simply connected window has *cylinder* topology, (as opposed to the separate absolute minimal-area orientable spanning surfaces S_{1L} and S_2 of W_{1L} and W_2), and secondly, that this cylinder-topology absolute minimal-area orientable spanning surface of the non-simply connected window, is given to a good approximation by the appropriate solution of Laplace's equation, in the projection onto S_{1L} of the non-simply connected window. Clearly, for our calculation to be correct, these assumptions must be valid for all (w, x, y, z) which give significant contributions to the integral over (w, x, y, z) of $J(w, x, y, z)$. Now of course, (7.19) only gets contributions from (w, x) in S_{1L} and not too close to the edge of S_{1L}, so we have to check that, for (w, x) in this domain, our assumptions are valid for all (y, z) that give significant contributions to (7.17).

We note, however, that we do *not* have to check that our assumptions are valid for *all* island configurations centred at (w, x, y, z): that is manifestly false, even for $(w, z, y, z) = (0, 0, 0, 0)$, and we did not assume that. What we have to check is that, for each (w, x, y, z) in the domain concerned, our assumptions are valid for *some* island configurations centred at (w, x, y, z). Thus it is sufficient to check, for each (w, x, y, z) in the domain concerned, that our assumptions are valid when W_2 is the circle specified before (7.10).

Now from (7.16) we immediately see that the main contributions to (7.17) come from (y, z) such that:

$$\left(y^2 + z^2\right) \leq \frac{1}{\mu^2 \pi} \ln\left(\mu L \left(1 - \frac{4\left(w^2 + x^2\right)}{L^2}\right)\right) \tag{7.20}$$

where we set $v = 1$ because its presence does not have any significant effect on the magnitude of the right-hand side of (7.20).

And from (7.11) - (7.13), we see that when W_2 is the circle specified before (7.10), the maximum value of $(\partial_\mu \tilde{y})(\partial_\mu \tilde{y}) + (\partial_\mu \tilde{z})(\partial_\mu \tilde{z})$, (in the notation of (7.14)), which is realized by the radial derivatives on W_2, is given by $\left(\frac{b}{r}\right)^2$ at $r = \frac{1}{2\mu}$, or in other words, (remembering that b is a 2-vector), by:

$$\frac{4\mu^2\left(y^2 + z^2\right)}{\left(\ln\left(\mu L\left(1 - \frac{4(w^2+x^2)}{L^2}\right)\right)\right)^2} \tag{7.21}$$

which by (7.20) is less than or equal to:

$$\frac{4}{\pi \ln\left(\mu L\left(1 - \frac{4(w^2+x^2)}{L^2}\right)\right)} \tag{7.22}$$

for all (y, z) which give the main contributions to (7.17). Now (7.22) is small compared to 1 for large μL, hence, since the corrections to the Laplace approximation (7.14) to the "out-of-plane" contributions to the area are of order $\left(\left(\partial_\mu \tilde{y}\right)\left(\partial_\mu \tilde{y}\right) + \left(\partial_\mu \tilde{z}\right)\left(\partial_\mu \tilde{z}\right)\right)^2$, (7.14) *does* give a good approximation to the "out-of-plane" contributions to the area, hence (7.10) does give a good approximation to the area of the "Laplace surface", for all (y, z) which give the main contributions to (7.17). Now by Courant's results [39], the absolute minimal-area orientable spanning surface will have cylinder topology if there *exists* an orientable spanning surface with cylinder topology whose area is less than the sum $\frac{\pi}{4}\left(L^2 + \frac{1}{\mu^2}\right)$ of the areas of the separate absolute minimal-area spanning surfaces of W_{1L} and W_2, so it comes down to whether (7.10) is less than or greater than $\frac{\pi}{4}\left(L^2 + \frac{1}{\mu^2}\right)$, and we see immediately from (7.20) that (7.10) is *less* than $\frac{\pi}{4}\left(L^2 + \frac{1}{\mu^2}\right)$, for all (y, z) that give the main contributions to (7.17). Thus for all (y, z) that give the main contributions to (7.17), there *do* exist island configurations of size $\frac{1}{\mu}$, centred at (y, z), such that the absolute minimal-area orientable spanning surface of the non-simply connected window has cylinder topology, and is moreover given, to a good approximation, by the appropriate solution of Laplace's equation, in the projection onto S_{1L} of the non-simply connected window.

Thus there is no escape: our assumptions have been valid, and our calculation has been correct. Formula (7.19) *does* give the leading term in the right-hand side of the group-variation equation for $f_0(W_{1L}, g^2)$ at large L, and, due to the logarithmic factor in (7.19), our ansatz does *not* satisfy the group-variation equation for $f_0(W_{1L}, g^2)$ in the area-law domain. What is the significance of this?

Let us first ask whether the behaviour suggested by (7.19) gives an acceptable modification to the Wilson area law. Multiplying (7.19) by $-\frac{2\beta(g)}{g}$, and assuming that K is negative, we see, by comparison with (5.7), that (7.19) gives an "output behaviour" of $f_0(W_{1L}, g^2)$ at large L, of the form:

$$a\, e^{-bL^2 \ln(cL)} \tag{7.23}$$

where a, b, and c are constants. This does *not* match the "input behaviour" $e^{-\mu^2 \frac{\pi}{4} L^2}$ of $f_0(W_{1L}, g^2)$ at large L. Does (7.23) give an acceptable modification of the Wilson area law? The answer is *no*: (7.23) violates the Seiler bound [16], [17], which says that for simple planar loops, $f_0(W, g^2)$ cannot fall off faster than $e^{-\mu^2 A}$ for some *fixed* μ, where A is the area of the spanning surface of W. Thus the area law for $f_0(W, g^2)$ must stand, and our ansatz must be modified in some other domain.

Now in fact it is obvious what the problem is: for (w, x) in S_{1L} and not too close to the edge of S_{1L}, the rate at which $J(w, x, y, z)$, as defined after (7.10), falls off for increasing y and z, as given by (7.16), *decreases* logarithmically with increasing L, due to the logarithmic factor in the denominator in the exponent of (7.16), which in turn is due to the logarithmic factor in the denominator of the term in (7.10) quadratic in y and z. In other words, for island configurations of size $\frac{1}{\mu}$, centred at (w, x, y, z), such that the absolute minimal-area orientable spanning surface of the non-simply connected window has cylinder topology, the "tightness" with which that cylinder-topology surface is able to "draw back" the island towards S_{1L}, gets smaller and smaller as L gets larger: for purposes of restricting the freedom of movement of the island in the directions y and z perpendicular to S_{1L}, the cylinder-topology surface effectively gets "slacker" as L gets larger.

7.4 Pre-exponential factors for the higher-topology terms in the ansatz

Now it is obvious, and confirmed explicitly below, that if, in point (i) of our ansatz, as presented after (5.3), we had confined attention totally to the *separate* absolute minimal-area orientable spanning surfaces $S_{1L}, \ldots, S_{nL}$ of $W_{1L}, \ldots, W_{nL}$, and allowed no consideration at all of the higher-topology surfaces, (i.e. with two or more "holes" per connected component), then our qualitative argument, as given before (5.5), would have been exactly right, and our ansatz *would* have satisfied the group-variation equations in the area-law domain as well as in the glueball saturation domain. Indeed, when we substitute this modified ansatz into the right-hand side of the group-variation equation for $f_0(W_{1L}, \ldots, W_{nL}, g^2)$, it is then the exponential fall-off of the glueball propagator that "pulls back" the island to the separate absolute minimal-area spanning surfaces $S_{1L}, \ldots, S_{nL}$, and the "tightness" of this is completely independent of the sizes of the W_{iL}'s.

Note added: another possibility, overlooked in the first version of this paper, is that each exponential factor in the ansatz that involves the area of a non-simply connected, connected component of the absolute minimal-area orientable spanning surface of $W_{1L}, \ldots, W_{nL}$, or in other words, a connected, absolute minimal-area orientable spanning surface, whose boundary is q of the W_{iL}'s, where $q \geq 2$, has associated with

it a *pre-exponential* factor $\frac{1}{(\ln(\mu^2 A))^{q-1}}$, analogous to the pre-exponential factors associated with the line segments in the absolute minimal-length spanning tree. When this modified ansatz is substituted into the right-hand side of the group-variation equations, then, since the logarithm is slowly varying at large L, the leading order calculation, at large L, would be just as before, but with the area A in the logarithm simply set equal to the area A of the left-hand side Wilson loop, which would then cancel the unwanted logarithmic factor in (7.19). As we will see below, in this case, the group-variation equation result for the Wilson area law coefficient, μ^2, sets μ^2 equal to the sum of (7.19) with twice the unwanted logarithmic factor divided out, plus the contribution, calculated below, from the term in the ansatz, (which, as modified, becomes a sum of terms, over the various possible topologies of the total absolute minimal-area orientable spanning surface), where, instead of the cylinder topology absolute minimal-area orientable spanning surface of W_{1L} and W_2, we take the two separate absolute minimal-area orientable spanning surfaces of W_{1L} and of W_2, with a minimal-length straight-line segment, corresponding to the lightest glueball propagator, connecing them. In this case, we do not get the zeroth-order result $m = 2.38\mu$ quoted in the introduction to the paper, because there is no obvious relation between the cylinder contribution and the "lightest glueball propagator" contribution. However, a third possibility is that the pre-exponential factor associated with the "higher-topology" absolute minimal-area orientable spanning surfaces gives a stronger suppression than the "minimal" possibility $\frac{1}{(\ln(\mu^2 A))^{q-1}}$, for example, higher powers of the logarithm in the denominator, or even powers of the area itself in the denominator, analogously to the pre-exponential factors for the straight line segments of the absolute minimal-length spanning tree. In this case, the leading contribution to the right hand side of the group-variation equations, at large L, in the "area-law domain", would come totally from the term in the ansatz where the island is connected to an absolute minimal-area orientable spanning surface of the left-hand side Wilson loops, by a straight segment, or in other words, a lightest-glueball propagator. In this case, we again get the zeroth-order relation $m = 2.38\mu$, as given in the first version of this paper, where "suppression factors" for the higher-topology terms were assumed, which, in fact, do not give the suppression required. Since the requirement of the self-consistency of the ansatz at long distances doesn't seem to choose between these two possibilities, the correct possibility must be determined by integrating the group-variation equations from short distances, with boundary conditions at short distances, (i.e. very small L), given by renormalization-group-improved

perturbation theory. In practice, the group-variation equations will be re-expressed as equations for the long-distance factors (5.38) of the Wilson loop vacuum expectation values and correlation functions, which will be subject to the boundary conditions that the vacuum expectation values $f_0(W_{1L}, g^2)$ tend to 1 as L becomes very small, and the correlation functions $f_0(W_{1L}, \ldots, W_{nL}, g^2)$, $(n \geq 2)$, tend to the values of the correlation functions, as determined by renormalization-group-improved perturbation theory, and with the vacuum expectation values of $W_{1L}, \ldots, W_{nL}$ divided out. We now return to the discussion in the first version of this paper.

Can we then completely abandon consideration of the higher-topology spanning surfaces? The answer would appear to be no: if we consider $f_0(W_{1L}, W_{2L}, g^2)$ when W_{1L} and W_{2L} are two large loops that track one another closely, but have opposite orientations, (i.e. their arrows point in opposite directions), then we must surely expect $f_0(W_{1L}, W_{2L}, g^2)$ to be determined by the area of the cylinder-topology surface, i.e. by the closed loop of "ribbon" that runs between W_{1L} and W_{2L}, rather than the much larger areas of the separate absolute minimal-area orientable spanning surfaces of W_{1L} and W_{2L}. Indeed, when two such loops track one another *very* closely, such that, if they are composed of straight segments, then throughout the length of each segment, the separation between the two paths is small compared to $\frac{1}{\mu}$, and *also* small compared to the length of that segment, then $f_0(W_{1L}, W_{2L}, g^2)$ is given to a good approximation by perturbation theory, as the exponential of $-\frac{g^2}{4\pi}$ times the integral, over *one* of the two paths, of the reciprocal of the shortest distance to the other path. This bears no relation at all to the areas of the separate absolute minimal-area spanning surfaces of the two loops, but can certainly make a natural transition, as the separation between the two loops increases, to a dependence on the area of the absolute minimal-area, cylinder-topology spanning surface of the two loops, i.e. the closed loop of "ribbon" which runs between the two loops.

Such pairs of loops, which track one another closeley, but have opposite orientations, are important in the *transition* from the asymptotic freedom domain to the area law domain.[2] Here we decouple the group-variation equation for $f_0(W_{1L}, g^2)$ from the other group-variation equations, by treating, in the island diagrams in the right-hand side of the group-variation equation for $f_0(W_{1L}, g^2)$, the non-simply connected window by perturbation theory, (i.e. substituting in the perturbative expansion of

[2]Note added: or rather, in the domain where the long-distance factors are just beginning to differ from their boundary conditions at very small L, as described just above.

$f_0(W_{1L}, W_2, g^2)$), while treating the *internal* windows of the island, (i.e. the simply connected windows), non-perturbatively in the usual way. In this case, when L is approximately equal to $\frac{1}{\mu}$, island configurations where W_2, the outer boundary of the island, closely tracks W_{1L}, but has the opposite orientation, are very important. In fact, if we consider just the one-loop islands, we obtain, to a good approximation, a simple first-order *ordinary linear* differential equation for $f_0(W_1 L, g^2)$, with only slight couplings between $f_0(W_{1L}, g^2)$ for loops in different "scaling families", and this results, as L increases towards $\frac{1}{\mu}$ from values small compared to $\frac{1}{\mu}$, in gradually exponentiating the initially weak dependence of $f_0(W_{1L}, g^2)$ on the onset point R of the smooth long-distance cutoffs we impose on the propagators in our counterterms, and this drives the transition to the area-law behaviour. Note added: the statement that we obtain, to a good approximation, an ordinary linear differential equation for $f_0(W_{1L}, g^2)$, when L is approximately equal to $\frac{1}{\mu}$, is probably not correct. What seems possible, however, is that, due to the soft dependence of the long-distance factors on the details of the path, there will be a region, starting at very small L, where the long-distance factors barely differ from their boundary conditions at small L, as described just above, and continuing to larger L, but ending *well before* L reaches a size of about $\frac{1}{\mu}$, where the long-distance factor for $f_0(W_{1L}, g^2)$ will approximately satisfy an ordinary first-order differential equation with respect to L, which will be linear if we consider only the one-loop islands. This is because, due to the soft dependence of the long-distance factor for the interior of the island on the details of the island perimeter, W_2, we may, in some approximation, in this region, replace the long-distance factor for the island, in regions where the perimeter W_2 of the island does not wander too far from the left-hand side Wilson loop W_{1L}, by the long-distance factor for the left-hand side Wilson loop W_{1L}, while when the perimeter of the island is not close to the left-hand side Wilson loop, we would treat the interior of the island, as well as the non-simply connected window, by renormalization-group-improved perturbation theory. We now return to the discussion in the original version of this paper. The "initial" value of $f_0(W_{1L}, g^2)$, at very small L, is determined, for each "scaling family" of loops W_{1L}, by perturbation theory. And as the area law starts to set in as L increases to values greater than $\frac{1}{\mu}$, the typical island size stops at $\frac{1}{\mu}$, and we begin to use the group-variation equation for $f_0(W_{1L}, W_{2L}, g^2)$, rather than perturbation theory, to determine the window weight for the non-simply connected window.

In general, if $W_{1L}, \ldots, W_{nL}$ are such that W_{1L} is a very large loop, and $W_{2L}, \ldots, W_{nL}$

are smaller loops, but all of them large compared to $\frac{1}{\mu}$, and $W_{2L}, \ldots, W_{nL}$ all lie within, or nearly within, the absolute minimal-area orientable spanning surface S_{1L} of W_{1L}, and are all oriented consistently with the orientation of S_{1L} defined by W_{1L}, (so that, in other words, if S_{1L} is drawn on a flat surface, and $W_{2L}, \ldots, W_{nL}$ are drawn on S_{1L}, then the handedness, clockwise or anti-clockwise, of the arrows on $W_{2L}, \ldots, W_{nL}$, is opposite to the handedness of the arrow on the boundary W_{1L} of W_{1L}), and the separations between the loops are all large compared to $\frac{1}{\mu}$, then we would expect $f_0(W_{1L}, \ldots, W_{nL}, g^2)$ to be given roughly by $e^{-\mu^2 A}$, where A is the area of the surface S obtained from S_{1L} by cutting out of it the separate minimal-area spanning surfaces $S_{2L}, \ldots, S_{nL}$ of $W_{2L}, \ldots, W_{nL}$, so that S is, indeed, the "higher-topology", *absolute* minimal-area orientable spanning surface of $W_{1L}, \ldots, W_{nL}$, exactly as specified in point (i) of our ansatz, as presented after (5.3). And we cannot simply dismiss such sets of loops as of no physical importance, since they are likely to be important, for example, in calculating the corrections to meson propagators from quark-antiquark vacuum bubbles. How can we reconcile this with the conclusion we arrived at after (7.23), namely that for $f_0(W_{1L}, W_{2L}, g^2)$, if W_{1L} is large compared to $\frac{1}{\mu}$, but the size of W_{2L} is roughly equal to $\frac{1}{\mu}$, then $f_0(W_{1L}, W_{2L}, g^2)$ is determined by the *separate* absolute minimal-area orientable spanning surfaces S_{1L} and S_{2L} of W_{1L} and W_{2L}, together with the glueball propagator for the shortest straight line segment between any point on S_{1L} and any point on S_{2L}, completely irrespective of whether or not the *absolute* minimal-area orientable spanning surface of W_{1L} and W_{2L} actually has cylinder topology?

Note added: the following two paragraphs discuss an example in QCD-2, and its relation to the four-dimensional case, which was intended to motivate the suppression factors for the higher-topology terms in the ansatz that were proposed in the original version of this paper. However, those suppression factors do not, in fact, give the suppression required, so the discussion of the QCD-2 example is, strictly speaking, no longer relevant.

A possible compromise to consider is suggested by QCD-2. Indeed, in QCD-2, with coupling constant μ, if W_{1L} and W_{2L} are simple planar loops, with no self-intersections and no mutual intersections, and W_{2L} lies inside W_{1L}, and their orientations are opposite to one another, (so that if W_{1L} is shrunk until it coincides with W_{2L}, then their arrows point in *opposite* directions), and the area of the non-simply connected windows, insied W_{1L} but outside W_{2L}, is A, and the area of the simply-connected window, inside W_{2L}, is B, (so that the *total* area enclosed by W_{1L} is $A + B$), then $f_0(W_{1L}, W_{2L}, \mu^2)$ is

given by:

$$f_0(W_{1L}, W_{2L}, \mu^2) = e^{-\mu^2 A}\left(1 - \left(1 + 2\mu^2 B\right) e^{-2\mu^2 B}\right) \tag{7.24}$$

Now A here is the area of the *absolute* minimal-area orientable spanning surface of W_{1L} and W_{2L}, which has cylinder topology, and for B large compare to $\frac{1}{\mu^2}$, the factor in brackets in the right-hand side of (7.24) becomes equal to 1, so that $f_0(W_{1L}, W_{2L}, g^2)$ becomes equal to $e^{-\mu^2 A}$, exactly as given by the initial form of our ansatz, as stated after (5.3). However, for *small* B, (7.24) becomes

$$2\mu^4 B^2 e^{-\mu^2 A} \tag{7.25}$$

and thus vanishes quadratically in B as B tends to zero. Now actually, this behaviour is completely expected. Any Wilson loop whose absolute minimal-area orientable spanning surface has zero area, is identically equal to 1: such a Wilson loop can be the trace of a zero-length path-ordered phase factor, or, more generally, the trace of a "tree" formed of hairpin-shaped path-ordered phase factors, which exactly double back on themselves. (Such "trees" of hairpin-shaped path-ordered phase factors are not, however, very important, since they are suppressed by the *kinematic* weights in path integrals, e.g. the Gaussian factors in (1.27). Note added: they might, however, be important in connection with chiral symmetry breaking, when a light quark moves along the perimeter of the loop, since in that case, they look like the emission of pions into the vacuum.) Thus any correlation function involving a Wilson loop whose absolute minimal-area orientable spanning surface has zero area, vanishes identically. This explains why (7.24) vanishes at zero B. And the reason (7.24) vanishes *quadratically* in B, rather than linearly in B, for small B, is due to the vanishing of the SU(N) group factor, for all N, for any Feynman diagram that can be separated into two disconnected parts, with W_{1L} in one part, and W_{2L} in the other part, by cutting exactly one propagator. (This is due, as explained after (1.53), to a cancellation between the two terms in (1.50) for that one "key" propagator.)

Now of course, although this "formal" argument for the vanishing of any correlation function involving any Wilson loop whose absolute minimal-area orientable spanning surface has zero area, works just as well in four dimensions as it does in two dimensions, there are, in four dimensions, competing tendencies for quantities to become singular as the sizes of Wilson loops tend to zero. Thus we did not make any allowance for these competing tendencies in our initial ansatz, as stated after (5.3), but rather left it for the group-variation equations to determine the correct behaviour. And in view of

our result above, namely that when W_{1L} is large compared to $\frac{1}{\mu}$, but the size of W_{2L} is roughly equal to $\frac{1}{\mu}$, $f_0(W_{1L}, W_{2L}, g^2)$ must be determined by the *separate* absolute minimal-area orientable spanning surfaces S_{1L} and S_{2L} of W_{1L} and W_{2L}, together with the glueball propagator for the shortest straight line segment between any point of S_{1L} and any point on S_{2L}, completely irrespective of whether or not the *absolute* minimal-area orientable spanning surface of W_{1L} and W_{2L} actually has cylinder topology, and our observation that when the sizes of *all* the loops are large compared to $\frac{1}{\mu}$, we *do* have to consider the possibility that the absolute minimal-area orientable spanning surface might have "higher topology", (i.e. fewer connected components, and more holes per connected component), and the above example from QCD-2, we now modify our initial ansatz, as stated after (5.3), as follows:

$f_0(W_1, \ldots, W_n, g^2)$ is now expressed as a *sum* of terms, each associated with a different topology of spanning surface of $W_1, \ldots, W_n$, or in other words, with a different partition of $\{W_1, \ldots, W_n\}$ into parts corresponding to the separate connected components of the spanning surface. And the term associated with a given topology of spanning surface, or in other words, with a given partition of $\{W_1, \ldots, W_n\}$ into parts corresponding t the separate connected components of the spanning surface, is given by the product, firstly, of exactly the same factors as before, (i.e. as stated after (5.3)), with the proviso that the area A in point (i) is now the area of the absolute minimal-area orientable spanning surface *with the given topology*, if it exists, and secondly, of possible additional factors, as follows. If *no* absolute minimal-area orientable spanning surface with the given topology exists, then the additional factor is zero. And if an absolute minimal-area orientable spanning surface with the given topology does exist, (the additional factors specified in the following, differ from those proposed in the original version of this paper), then the additional factors consist of the product, over all parts j of the given partition of $\{W_1, \ldots, W_n\}$ whose number q_j of members is ≥ 2, of a factor

$$\Big(\mathbf{F}(\mu^2 A_j)\Big)^{q_j - 1} \tag{7.26}$$

where A_j is the area of the absolute minimal-area, oriented, *and connected*, spanning surface of the $q_j \geq 2$ members of the j^{th} part of the given partition of $\{W_1, \ldots, W_n\}$, and $\mathbf{F}(s)$ is a fixed real function, which is either, in the "marginal" case, equal to $\frac{1}{\ln(s)}$, or, in the "non-marginal" cases, equal to a product of powers of $\ln(s)$ and s, which is small compared to $\frac{1}{\ln(s)}$, as $s \to +\infty$. As we will see, the requirement of self-consistency of the ansatz at large distances does not appear to distinguish between the "marginal"

case, and the "non-marginal" cases, so determining the explicit form of $\mathbf{F}(s)$, as either the "marginal case" possibility, or one of the "non-marginal" possibilities, will require integrating the group-variation equations from small L, with the boundary conditions at small L as discussed above.

If it were not for the extra factors (7.26), the behaviour of our modified ansatz would essentially be the same as the behaviour of the original ansatz, as stated after (5.3), since the original ansatz essentially consists of picking out whichever of the terms of our modified ansatz, without the extra factors, is largest. But the extra factors (7.26) perform the crucial function of suppressing the contributions of terms corresponding to "higher topology" spanning surfaces, (i.e. with fewer connected components, and more holes per connected component), such that they now either give the correct form of the leading contribution to the right-hand sides of the group-variation equations in the "area-law" domain, (in the "marginal case"), which is the same form as given by the relevant term in the ansatz where the island is connected to a connected component of a minimal-area spanning surface of the left-hand side Wilson loops, by a straight-line segment, or "lightest glueball propagator", or else give a contribution to the right-hand sides of the group-variation equations in the "area-law" domain, (in the "non-marginal" cases), which is small compared to the leading term, which is now given totally by the relevant term in the ansatz where the island is connected to a connected component of a minimal-area spanning surface of the left-hand side Wilson loops, by a straight-line segment, or "lightest glueball propagator".

Thus when we substitute the modified ansatz into the right-hand side of the group-variation equation for $f_0(W_{1L}, \ldots, W_{nL}, g^2)$, $n \geq 1$, the leading terms at large L, which come from island diagrams, with islands of size roughly $\frac{1}{\mu}$, come, in the *non-marginal* cases, from terms in our ansatz, where any boundaries of the island that form parts of the borders of non-simply connected windows, belong to *one-member* parts of the partitions of the sets of the connected components of the borders of those non-simply connected windows, that define those terms in our ansatz for those non-simply connected windows. The contributions of terms in our ansatz where boundaries of the island are involved in "higher topology" spanning surfaces of the boundaries of those non-simply connected windows, are suppressed, in the *non-marginal* cases, by the extra factors (7.26). In the *marginal* case, the leading terms in the right-hand side come from the terms just specified, *plus* the terms that give the leading contribution for the non-modified ansatz, as calculated in (7.19), which is now modified to have the correct

form, by dividing by twice the logarithmic factor.

We can now re-do, with our modified ansatz, the calculation of the leading term, at large L, in the contribution of any island diagram to the right-hand side of the group-variation equation for $f_0(W_{1L}, g^2)$. We consider the *non-marginal* cases, since the result for the marginal case can be obtained from the result for the *non-marginal* cases, simply by adding (7.19), with twice the logarithmic factor divided out. It is obvious that, since the leading term in our ansatz for $f_0(W_{1L}, W_2, g^2)$, where W_2 is the outer boundary of the island, is now:

$$f^2 e^{-\mu^2 A_{1L}} e^{-\mu^2 A_2} \sqrt{\frac{m}{32\pi^3 |x-y|^3}} e^{-m|x-y|} \tag{7.27}$$

where f is the glueball to surface coupling constant introduced in point (iii) of our original ansatz, as stated after (5.3), A_{1L} is the area of the absolute minimal-area orientable spanning surface S_{1L} of W_{1L}, A_2 is the area of the absolute minimal-area orientable spanning surface S_2 of W_2, and x and y are the ends, on S_{1L} and S_2 respectively, of the shortest straight-line segment between any point on S_{1L} and any point on S_2, we now get exactly the right dependence on L, namely, within a given "scaling family" of loops W_{1L}, a dependence of the form:

$$aL^2 f_0(W_{1L}, g^2) \tag{7.28}$$

where a is a constant.

Indeed, we now recall that, as discussed after (7.3), every Type-1 island diagram is obtained from some connected, one-line irreducible vacuum bubble formed of 45-paths, that may be drawn on the surface of the 2-sphere without any 45-paths crossing one another, by cutting n holes, (where $n = 1$ in the present case), in *one* of the windows of that vacuum bubble, and stretching that window to form the non-simply connected window that "surrounds" the island. And we also recall that, before (7.4), we defined, for each such vacuum bubble b drawn on the 2-sphere, X_b to be the result of doing the path integrals over all the 45-paths of b, with a window weight $e^{-\mu^2 B_i}$ for each window i of b, (where B_i is the area of the absolute minimal-area orientable spanning surface of the boundary of window i of b), subject to the mean position of all the vertices in all the 45-paths of b, having the fixed value z. We recall that, by translation invariance, X_b is independent of z, and we recall that, by definition, X_b *includes* any symmetry factor for b, (such as the symmetry factor $\frac{1}{2}$ for the vector-boson one-loop vacuum bubble),

and that, also by definition, X_b includes the standard factor $\frac{\mathrm{d}}{\mathrm{d}M}\mathbf{C}(M)\Big|_{M=1}$, which is completely unaffected by making holes in the windows of b.

We also recall that, for each of our vacuum bubbles b, we define n_b to be the number of windows of b.

Let us now, for simplicity, make the assumption, that the mass m of the lightest glueball, is sufficiently smaller than the mass M of the lowest-mass state of the cylinder with two 45-paths along it, that there is no significant tendency, for a fixed mean position z of all the vertices of the 45-paths of the island, for the glueball propagator between x and y, (where x and y are the ends, on S_{1L} and S_2 respectively, of the shortest straight line segment between any point of S_{1L} and any point of S_2), to "pull" part of the island towards S_{1L}. This assumption is precisely analogous to the assumption which we made, also for simplicity, in studying the glueball saturation domain, that m is sufficiently smaller than M, that there is no significant tendency for an island to elongate along the straight line segment of the minimal-length spanning tree that it is close to.

It follows immediately from this assumption, that in (7.27), instead of taking y strictly as the closest point, on S_2, to any point of S_{1L}, we may, to a good approximation, set y simply equal to any chosen vertex of W_2, or maybe to the mean position of all the vertices of W_2. We then find that, for the given values of x and y, the contribution of this island diagram to the right-hand side of the group-variation equation for $f_0(W_{1L}, g^2)$, is equal to:

$$f^2 e^{-\mu^2 A_{1L}} \sqrt{\frac{m}{32\pi^3 |x-y|^3}} e^{-m|x-y|} \tag{7.29}$$

times the path integrals over all the 45-paths of the island, with a window weight $e^{-\mu^2 B_i}$ for each window i of the island, (*including* the outer boundary window W_2), subject to the chosen vertex of W_2, (or, if preferred, the mean position of all the vertices of W_2), having the given value y, times the standard factor $\frac{\mathrm{d}}{\mathrm{d}M}\mathbf{C}(M)\Big|_{M=1}$ for that island diagram, times a possible symmetry factor, associated with rotational symmetries of the island. Here B_i is the area of the absolute minimal-area orientable spanning surface of the boundary of the window i of the island, and the factor $e^{-\mu^2 B_i}$ for the outer boundary W_2 of the island, is the factor $e^{-\mu^2 A_2}$ in (7.27). But by the translation invariance of the sums over the 45-paths of the island, and of the window weights B_i, this is precisely equal to (7.29), times X_B, where b is the vacuum bubble corresponding to that island, times a possible integer factor, associated with symmetries

of b, equal to the number of different windows of b, such that making a hole in that window, gives the given island diagram. (The integer factor, if it occurs, is equal to the symmetry factor assoicated with any rotational symmetries of the island, divided by the symmetry factor associated with the vacuum bubble b.) We note that, if the vacuum bubble b has two or more loops, then X_b, and also the corresponding island diagrams, also include an explicit power of g^2, equal to the number of 45-paths minus the number of action vertices of that vacuum bubble or island. This power of g^2 is in general one less than the number of loops of that vacuum bubble or island. We also note that the factor $e^{-\mu^2 A_{1L}}$ in (7.29), is equal to $f_0(W_{1L}, g^2)$.

We thus find immediately that, subject to our assumption, made for simplicity, that m is sufficiently smaller than M, that we may, to a good approximation, set y, in (7.27), equal to the mean position of the vertices of W_2, the contribution of all the island diagrams, when we substitute our modified ansatz into the right-hand side of the group-variation equation for $f_0(W_{1L}, g^2)$, is equal to $f^2 \left(\sum_b n_b X_b \right) f_0(W_{1L}, g^2)$, times the integral:

$$\int_{S_{1L}} \mathrm{d}^2 x \int \mathrm{d}^2 y \sqrt{\frac{m}{32\pi^3 \left| x-y \right|^3}} e^{-m|x-y|} \tag{7.30}$$

since of course, as L becomes large, and correspondingly, both radii of curvature of S_{1L} tend to zero at all points of S_{1L}, x becomes simply equal to the perpendicular projection onto S_{1L} of the mean position y of the vertices of the island, and furthermore, again due to the radii of curvature of S_{1L} tending to zero as L becomes large, and also due to the suppression of the integrand in (7.29) and (7.30) when $|x-y|$ is large compared to $\frac{1}{m}$, we may, for large L, represent the four-dimensional integral over the mean position y of the vertices of the island, as the two-dimensional integral, over S_{1L}, of the position of the perpendicular projection x of y onto S_{1L}, times the two-dimensional integral, over all y such that the perpendicular projection of y onto S_{1L}, (or more precisely, onto the two-plane tangential to S_{1L} at x), is equal to x. (The two-dimensional integral over y in (7.30) runs over the two dimensions perpendicular to the two-plane tangential to S_{1L} at x.)

Now, setting $z \equiv y - x$, the two-dimensional integral over y in (7.30) is simply:

$$\int \mathrm{d}^2 z \sqrt{\frac{m}{32\pi^3 \left| z \right|^3}} e^{-m|z|} = 2\pi \sqrt{\frac{m}{32\pi^3}} \int_0^\infty \frac{\mathrm{d}r}{\sqrt{r}} e^{-mr} = \frac{1}{2\sqrt{2}} \tag{7.31}$$

and the two-dimensional integral over x gives simply A_{1L}. Hence we find that, for large L, the leading term in the sum over all the island diagrams in the right-hand side of the

group-variation equation for $f_0(W_{1L}, g^2)$, and, indeed, the leading term in the sum over *all* the diagrams in the right-hand side of the group-variation equation for $f_0(W_{1L}, g^2)$, is equal, for all the possible non-marginal forms of the pre-exponential function $\mathbf{F}(s)$) in (7.26), to:

$$\frac{1}{2\sqrt{2}} A_{1L} f^2 \left(\sum_b n_b X_b \right) f_0(W_{1L}, g^2) \tag{7.32}$$

Hence, multiplying by $-\frac{2\beta(g)}{g}$, and comparing with (5.7) and (5.8), and noting that $a_1 L^2$, in (5.8), is equal to A_{1L} in our present notation, we see that our modified ansatz does indeed exactly satisfy the group-variation equation for $f_0(W_{1L}, g^2)$ at large L, and furthermore, that the Wilson area-law parameter μ^2 is given, for all the possible non-marginal forms of the pre-exponential function $\mathbf{F}(s)$) in (7.26), by:

$$\mu^2 = \frac{1}{2\sqrt{2}} \frac{\beta(g)}{g} f^2 \left(\sum_b n_b X_b \right) \tag{7.33}$$

This is the equation that replaces (5.10) for our modified ansatz, for all the possible non-marginal forms of the pre-exponential function $\mathbf{F}(s)$) in (7.26).

(We of course assume, as usual, that $\left(\sum_b n_b X_b \right)$ is *negative*, and, indeed, that the sum of X_b for the one-loop vacuum bubbles, corresponding to the island diagrams (4.25), (4.26), and (4.27), is *negative*.) We see that (7.33), which we have now derived quantitatively from our modified ansatz, for all the possible non-marginal forms of the pre-exponential function $\mathbf{F}(s)$) in (7.26), (subject to our assumption, made for simplicity, that the mass m of the lightest glueball, is sufficiently smaller than the mass M of the lowest-mass state of the cylinder with two 45-paths along it, that there is no significant tendency, for a fixed mean position of all the vertices of the island, for the glueball propagator to "pull" part of the island towards S_{1L}), is essentially identical in structure to (5.10), apart from the presence of the factor f^2, where f is the glueball to surface coupling constant, introduced in point (iii) of our original ansatz, as stated after (5.3).

Considering now the marginal case, where the pre-exponential function $\mathbf{F}(s)$) in (7.26) has the form $\frac{1}{\ln s}$, it is necessary, first, to determine whether the leading term (7.19), in the contributions of the higher-topology terms in the ansatz, to the right-hand sides of the group-variation equations, (before multiplying by the pre-exponential factor $\mathbf{F}(\mu^2 A)$, where A is the area of the higher-topology minimal-area spanning surface involved, which is equal, to sufficient accuracy for the pre-exponential factor, in the

leading term, to the area of the minimal-area spanning surface of the appropriate left-hand side Wilson loops), has the same form, with the same coefficient, irrespective of the shapes of the left-hand side Wilson loops, or whether its form, or coefficient, which has so far been derived only for a circular left-hand side Wilson loop, depends on the shapes of the left-hand side Wilson loops, because if its form, or coefficient, *did* depend on the shapes of the left-hand side Wilson loops, then we could eliminate the marginal possibility for $\mathbf{F}(s)$, as being inconsistent with the Wilson area law, for large L.

However, it appears that, at least for planar left-hand side Wilson loops, even with self-intersections, the coefficient of $\ln(\mu L)$ in (7.19) is in fact completely independent of the shape of the left-hand side Wilson loop. Indeed, repeating, for a general planar left-hand side Wilson loop, the analysis that led to (7.19), we would take, as the appropriate solution of Laplace's equation, to sufficient accuracy:

$$b\left(\ln r + \Phi(W_{1L}, \tilde{w}, \tilde{x}, w, x)\right) \tag{7.34}$$

where $r \equiv \sqrt{(w-\tilde{w})^2 + (x-\tilde{x})^2}$, (w, x) are coordinates in the region of the two-plane, bordered by parts, or the whole, of the left-hand side Wilson loop, in which the projection into the two-plane of the centre of the circle of radius $\frac{1}{2\mu}$ lies, $(\tilde{w}, \tilde{x})$ are the coordinates of the projection into the two-plane of the centre of the circle, and b is a two-vector coefficient, (in the two directions perpendicular to the two-plane defined by the left-hand side Wilson loop), to be determined. $\Phi(W_{1L}, \tilde{w}, \tilde{x}, w, x)$ is the solution of Laplace's equation, in the coordinates (w, x), regular in the closed region of the two-plane, bordered by parts, or the whole, of the left-hand side Wilson loop, that contains the point $(\tilde{w}, \tilde{x})$, and which takes the value $-\ln r = -\ln(\sqrt{(w-\tilde{w})^2 + (x-\tilde{x})^2})$ on the border of that domain, so that (7.34) vanishes exactly on the border of that domain.

We then require that:

$$b\left(\ln\left(\frac{1}{2\mu}\right) + \Phi(W_{1L}, \tilde{w}, \tilde{x}, \tilde{w}, \tilde{x})\right) = (y, z) + \text{ order}\left(\frac{1}{\mu L}\right) \tag{7.35}$$

where (y, z) are the coordinates of the centre of the circle of radius $\frac{1}{2\mu}$, in the two directions perpendicular to the two-plane, which fixes

$$b = \frac{(y, z)}{\left(\ln\left(\frac{1}{2\mu}\right) + \Phi(W_{1L}, \tilde{w}, \tilde{x}, \tilde{w}, \tilde{x})\right)} + \text{ order}\left(\frac{1}{\mu L}\right) \tag{7.36}$$

Now $\Phi(W_{1L}, \tilde{w}, \tilde{x}, w, x)$ has no dependence at all on μ, hence, since all lengths, describing the shape of W_{1L}, can be expressed as dimensionless multiples of L, the

only way the argument of the logarithm, in the denominator of (7.36), can become dimensionless, as it must, is for $\Phi(W_{1L}, \tilde{w}, \tilde{x}, \tilde{w}, \tilde{x})$, to contain a term $-\ln\left(\frac{L}{2}\right)$. Thus b tends, at large L, to the two-vector:

$$\frac{-(y,z)}{\ln(\mu L)} \tag{7.37}$$

exactly as for the case when W_{1L} is circular. Furthermore, the leading contribution to (7.14), which again comes entirely from the circle of radius $\frac{1}{2\mu}$, since (7.34) vanishes on the outer boundary, is

$$\pi b^2 \ln(\mu L) \to \pi \frac{y^2+z^2}{\ln(\mu L)} \tag{7.38}$$

exactly as for the case when W_{1L} is circular.

That the $-\ln\left(\frac{L}{2}\right)$ term in $\Phi(W_{1L}, \tilde{w}, \tilde{x}, \tilde{w}, \tilde{x})$ really does occur, as stated, can be checked explicitly, for example, in the extreme case where the "loop" W_{1L} becomes the pair of lines $w = \frac{L}{2}$ and $w = -\frac{L}{2}$, in which case we find:

$$\begin{aligned}\Phi(W_{1L}, \tilde{w}, \tilde{x}, w, x) \quad &= \quad -\ln\left(\frac{L}{2}\right) + \\ &+ \int_0^\infty \frac{\mathrm{d}\alpha}{\alpha} e^{-\frac{\alpha L}{2}} \left\{ \left(\frac{\cosh(\alpha\tilde{x})\cosh(\alpha x)}{\cosh\left(\frac{\alpha L}{2}\right)} + \frac{\sinh(\alpha\tilde{x})\sinh(\alpha x)}{\sinh\left(\frac{\alpha L}{2}\right)} \right) \cos(\alpha(w-\tilde{w})) - 1 \right\}\end{aligned} \tag{7.39}$$

Furthermore, the arguments regarding the decoupling of the integrals over the shape of the island, from the integral over the mean position (w, x, y, z) of the vertices of the island, go through exactly as before, and we again arrive at (7.19).

Furthermore, since, by (7.20), the main contributions to (7.17) come from (y, z) of magnitude less than or equal to

$$\frac{1}{\mu}\sqrt{\frac{\ln(\mu L)}{\pi}} \tag{7.40}$$

while for non-planar W_{1L}, the radii of curvature of the minimal-area orientable spanning surface of W_{1L} will be of order L, we expect that, for non-planar loops, we may, in a zone of thickness (7.40) about the minimal-area orientable spanning surface of W_{1L}, again choose coordinates where (w, x) represents the position of the perpendicular projection of the mean position of the vertices of the island, onto the minimal-area orientable spanning surface of W_{1L}, and (y, z) represents the position of mean position of the vertices of the island, in the directions perpendicular to the two-plane tangential, at (w, x), to the minimal-area orientable spanning surface of W_{1L}, and that we will get the same result, (7.19), again, for the leading term.

Thus the marginal possibility for the pre-exponential factor $\mathbf{F}(s)$ also appears to be compatible with the Wilson area law, and in this case, formula (7.33), for the Wilson area-law parameter μ^2, must be modified, by the addition of (7.19), with twice the logarithmic factor divided out. It will be necessary to integrate the group-variation equations from small L, with boundary conditions, at small L, as given by renormalization-group-improved perturbation theory, in order to determine whether the marginal case occurs or not.

Now comparison of points (i), (ii), and (iii) of our original ansatz, as stated after (5.3), shows immediately that the dimension of f is *length*, so that the dimension of $\frac{1}{f}$ is *mass*. It then follows immediately, by reasoning exactly analogous to that which gave (6.2) and (6.3), that $\frac{1}{f}$ obeys precisely the same renormalization group equation as the Wilson area law parameter μ, i.e. (6.3), with μ replaced by $\frac{1}{f}$. The dependence of f on our input parameter R, (the onset point of the smooth long-distance cutoffs we impose on the propagators in our counterterms), is simply through an overall factor R, and the dependence of $\frac{1}{f}$ on our input parameter g, is through exactly the same factor that gives the dependence of μ on g. Thus the product $f\mu$ is an absolutely fixed real number, completely independent of R and g, which we may expect to be of order 1, (and, of course, calculable from the group-variation equations).

Furthermore, on dimensional grounds, each X_b is equal to μ^4, times the *explicit* power of g^2 associated with that vacuum bubble, (i.e. a power of g^2 equal to the number of 45-paths minus the number of action vertices of that vacuum bubble), times a real number independent of R and g^2. And of course, the power of g^2 is zero for the one-loop vacuum bubbles. Hence, for the non-marginal cases, we may divide μ^2 out of (7.33), to obtain an equation for the critical value of g^2, (which applies throughout the domain where our ansatz applies), exactly as we obtained before from (5.10), (as discussed between (5.28) and (5.29), and after (5.37)). If the marginal case applies, we would expect an analogous result to hold. (Note that, in the marginal case, $\mathbf{F}(s)$ will have the form $c\frac{1}{\ln s}$, where c is a number that will need to be determined by integrating the group-variation equations from small L, with boundary conditions as given by renormalization-group-improved perturbation theory.)

And furthermore, bearing in mind that for the one-loop vacuum bubbles b, X_b is completely independent of g^2, we may expact that, exactly as before, the critical value of g^2 will essentially be determined by the point where $\frac{\beta(g)}{g}$ reaches a critical value, as determined by (7.33), (or its analogue, if the marginal case applies), with μ^2 divided

out. Hence, exactly as before, (in the discussion after (6.4)), we may expect that, if, in a natural renormalization scheme, (where the behaviour of the expansion coefficients in $2g\beta(g)$ is *no better* than the behaviour of the expansion coefficients in the expansions of other physical quantities in the *explicit* powers of g^2 that multiply the right-hand side group-variation equation diagrams), the trend shown by the first two terms in (6.4) continues, and all the expansion coefficients in $2g\beta(g)$ have the *same* sign, (i.e. *negative*), then, due to the fact that, in that case, $-2g\beta(g)$ will tend to infinity as g^2 approaches the radius of convergence of the expansion of $-2g\beta(g)$ in powers of g^2, the critical value of g^2, as determined by (7.33), (or its analogue, if the marginal case applies), will be *strictly less* than the radius of convergence of the expansion of $-2g\beta(g)$ in powers of g^2, and consequently, due to the assumption that the renormalization scheme is natural, the critical value of g^2 will also be *strictly less* than the radii of convergence of the expansions of other physical quantities in the explicit powers of g^2 that multiply the right-hand side group-variation equation diagrams. Hence, exactly as before, we may expect that, if all the expansion coefficients in $2g\beta(g)$ are negative in such a natural renormalization scheme, then the expansions of all physical quantities in the explicit powers of g^2 that multiply the right-hand side group-variation equation diagrams, will converge geometrically for all g^2 less than or equal to the critical value.

7.5 The zeroth-order value of $m_{0^{++}}/\sqrt{\sigma}$ when the pre-exponential factor is non-marginal

Now comparing (7.33) with (7.9), we see that, if the pre-exponential factor $\mathbf{F}(s)$ is non-marginal, the ratio $\frac{m^2}{\mu^2}$ is given by:

$$\frac{m^2}{\mu^2} = \frac{2\sqrt{2}\left(\sum_b n_b^2 X_b\right)}{\left(\sum_b n_b X_b\right)} \tag{7.41}$$

We now observe that the island diagram mechanism by which the Wilson area law, and massive glueball saturation, arise in the group-variation equations, works in exactly the same way in leading order, i.e. if we consider just the one-loop island diagrams, as it does in all orders. The higher-loop islands will correct the details, but they are not necessary for quark confinement.

Furthermore, as we have just noted, if all the expansion coefficients in $2g\beta(g)$ are negative in a natural renormalization scheme, then we may expect that the expansions of all physical quantities in the explicit powers of g^2 that multiply the right-hand side group-variation equation diagrams, converge geometrically for all g^2 less than or equal to the critical value. Indeed, as we noted in the discussion after (6.4), if the two terms displayed in (6.4) are the first two terms in a geometric series, then the radius of convergence is given by $\frac{g^2}{4\pi} = \frac{11\pi}{17} = 2.03$, while, as we noted in the discussion after (5.37), the largest value of our $\frac{g^2}{4\pi}$ for which there is concrete experimental evidence, (based of $\alpha_s = 0.3$ for charmonium), is $\frac{g^2}{4\pi} = 0.35$. Thus there is ample room for the expansions of all physical quantities in the explicit powers of g^2 that multiply the right-hand side group-variation equation diagrams, to have a convergence factor of $\frac{1}{2}$, or even better.

Thus we may obtain a good first approximation to the numerical value of $\frac{m}{\mu}$, if the pre-exponential factor $\mathbf{F}(s)$ is non-marginal, by restricting the sums over the vacuum bubbles b in the numerator and denominator of (7.41), to the one-loop vacuum bubbles. We then find immediately, since $n_b = 2$ for each one-loop vacuum bubble, (corresponding to the two fundamental representation Wilson loop "sides" of the closed-loop 45-path), that if the pre-exponential factor $\mathbf{F}(s)$ in (7.26) is non-marginal, then the leading approximation, in large-N_c QCD, to the mass m of the lightest glueball, in terms of the Wilson area law parameter μ, is given by:

$$m = \sqrt{4\sqrt{2}}\mu = 2.38\mu \tag{7.42}$$

(In checking (7.42), we should remember that m^2 gets equal contributions from corresponding Type-1 and Type-2 one-loop island diagrams, while μ^2 only gets contributions from Type-1 island diagrams.)

Now as we noted between (4.90) and (4.91), the experimental value of μ is 0.41 GeV, hence we find $m = 0.98$ GeV.

We note that, if we estimate the mass M of the lowest mass state of the cylinder with two 45-paths along it as twice the effective mass 1.3μ of a 45-path, then our assumption that m is strictly less than M is satisfied, but only just. Could this indicate that the mass of the second lightest glueball is close above m? Will "hyperfine" corrections, (i.e. $F_{\mu\nu}$ insertions), destabilize this estimate of M?

Equation (7.42) suggests that our additional assumptions, made for convenience, that m is sufficiently small compared to M, that glueball propagators do not sig-

nificantly "pull islands out of shape", will require further investigation. This raises a quesion about the accuracy of (7.42), but does not affect the verifications of the Wilson area law and massive glueball saturation, which do not depend on those assumptions. (Note that, as discussed after (4.90), 1.3μ is essentially the *smallest* reasonable estimate of the effective mass of a 45-path.)

The best lattice value of $\frac{m}{\mu}$ is 3.56 [4], so the zeroth-order estimate (7.42), which applies if the pre-exponential factor $\mathbf{F}(s)$ in (7.26) is non-marginal, is about 33 percent smaller than the best lattice value.

Now as discussed after (6.4), there is evidence, from 't Hooft's studies of planar diagrams [6], [7], that the behaviour of the sums of diagrams in the right-hand sides of the group-variation equations, considered as an expansion in the *explicit* powers of g^2 that multiply those diagrams, will be geometric, at worst. Since the zeroth-order estimate (7.42), which applies if the pre-exponential factor $\mathbf{F}(s)$ in (7.26) is non-marginal, corresponds to dropping all terms in the numerator and denominator of (7.41) with $n_b > 2$, we can use the lattice result to estimate what the convergence factor will have to be, if the pre-exponential factor $\mathbf{F}(s)$ is non-marginal.

As defined before (7.4), n_b is the number of windows of the vacuum bubble b, and the number of *explicit* powers of g^2 multiplying the corresponding island diagram, is $(n_b - 2)$. Let us suppose, for purposes of determining what the convergence factor would be, that

$$\sum_{b|n_b=n} X_b = X\alpha^{n-2} \tag{7.43}$$

where α is a convergence factor to be determined. Then (7.41), which applies if the pre-exponential factor $\mathbf{F}(s)$ in (7.26) is non-marginal, becomes:

$$\frac{m^2}{\mu^2} = \frac{2\sqrt{2}\sum_{n=2}^{\infty} n^2\alpha^{n-2}}{\sum_{n=2}^{\infty} n\alpha^{n-2}} = 2\sqrt{2}\frac{4-3\alpha+\alpha^2}{(1-\alpha)(2-\alpha)} \tag{7.44}$$

which, according to the best lattice result, is to equal $(3.56)^2$. This implies that α is equal to 0.5919, which is not much larger than the figure of $\frac{1}{2}$ suggested above. With this value of α, we can study the convergence of $\frac{m}{\mu}$, as the sums in the numerator and denominator of (7.44) are truncated at various values of n, and we find that going to n, (or n_b), $= 3$ gives a 26 percent error in $\frac{m}{\mu}$, $n = 4$ gives a 20 percent error, $n = 5$ a 15 percent error, $n = 7$ an 8 percent error, $n = 13$ a 1 percent error, and $n = 18$ a 0.1

percent error. After that, each extra decimal place of accuracy requires increasing n, or n_b, by 5.

Looking at examples such as (4.25), (4.30), (4.38), and (4.39), suggests that the chromatic polynomial factor $\frac{\mathrm{d}}{\mathrm{d}M}\mathbf{C}(M)\Big|_{M=1}$ included in X_b, as defined before (7.4), tends to alternate in sign as n_b increases, hence, bearing in mind the other points discussed in connection with the signs of the island diagram contributions, such as in arriving at (6.40), for example, and the fact that, as just noted, if the pre-exponential factor $\mathbf{F}(s)$ in (7.26) is non-marginal, then the convergence factor α must be strictly positive, it is clear that the signs of the island diagram contributions will require very careful study.

Chapter 8

Concluding Remarks

8.1 Dimensional Regularization

It would be nice to be able to study the Group-Variation Equations within dimensional regularization [18], [19], [20], in view of the large body of existing work within renormalization-group-improved perturbation theory, including such results as the β-function to four loops in $\overline{\rm MS}$ [9], which appears to be both compatible with, and very useful for, the Group-Variation Equations.

If so, it will be necessary to face the fact that not only the internal variables of diagrams, but also the observable physical quantities, namely the vacuum expectation values and correlation functions of Wilson loops, must be defined in d dimensions, with d complex. Indeed, these vacuum expectation values and correlation functions provide the window weights for the diagrams in the right-hand sides of the Group-Variation Equations, and not only the vertices in these diagrams, but also the strings of vertices along the paths, in the sums over paths, must be freely moveable in d dimensions, in order to obtain the correct d-dimensional propagators, for the leading terms, when the window weights are expanded in powers of g^2, for example.

Since we will divide the vacuum expectation values and correlation functions by short-distance factors, that remove the linear divergences, and soften the dependence on the fine details of the path, and re-write the Group-Variation Equations as equations for the ratios (5.38), restoring the short-distance factors perturbatively in the windows of the Group-Variation Equation right-hand side diagrams, by means of effective fields, it will be sufficient to study the ratios (5.38), the long-distance factors, for Wilson loops that consist of finite numbers of straight segments. The long-distance factor

of a vacuum expectation value, or correlation function, of Wilson loops defined on a total of n straight segments, will be, in d dimensions, d complex, a function of the $\frac{1}{2}n(n-1)$ independent distances between the n vertices at the ends of the segments. It will have symmetries generated by independent cyclic permutations of the vertices around the separate Wilson loops involved, and a two-fold symmetry generated by the simultaneous reversal of the orientation of all the Wilson loops involved. Let us denote such a long-distance factor by $F(r_{12}, \ldots, r_{(n-1)n}, g^2, d)$, where r_{ij}, $1 \leq i < j \leq n$, are the distances between the n vertices.

8.1.1 Inequalities that might be satisfied by the distances among finite sets of points, for arbitrary complex d

We do not have a model for a d-dimensional Euclidean space, for general finite, complex, d, although an interesting construction has been proposed recently in reference [63]. Therefore it is not really obvious that, when d is complex, we can even assume that the distances r_{ij} are real, in the "physical" domain, over which we integrate, and where we consequently have to calculate $F(r_{12}, \ldots, r_{(n-1)n}, g^2, d)$. However while, in the process of solving the Group-Variation Equations, by iterative substitution of the left-hand sides into the right-hand sides, for example, we will inevitably have to calculate $F(r_{12}, \ldots, r_{(n-1)n}, g^2, d)$ for complex d, the target of the calculation is to determine $F(r_{12}, \ldots, r_{(n-1)n}, g^2, 4)$, with all r_{ij} real. Therefore it might be reasonable to assume that, even at intermediate stages in the calculation, where we have to allow d to be complex, we can choose the "integration contours", in the d-dimensional complex Euclidean space, such that the r_{ij} are real.

Determining the domain of the $\{r_{ij}\}$, for which we actually have to calculate $F(r_{12}, \ldots, r_{(n-1)n}, g^2, d)$, at intermediate stages in the calculation, is very important, because we will not, in general, be able to find explicit, exact formulae for $F(r_{12}, \ldots, r_{(n-1)n}, g^2, d)$. What will, in practice, go into the right-hand sides of the Group-Variation Equations, and come out of the left-hand sides, will be various upper and lower bounds on the real and complex parts of $F(r_{12}, \ldots, r_{(n-1)n}, g^2, d)$, in various limited domains of the r_{ij}, g^2, and d. The process of solving the Group-Variation Equations will involve finding successively tighter sets of upper and lower bounds, on the real and complex parts of $F(r_{12}, \ldots, r_{(n-1)n}, g^2, d)$, in the domain of the $\{r_{ij}\}$ that actually occurs in the integrals in the right-hand sides of the Group-Variation Equations, such

that when we substitute these bounds into the right-hand sides of the Group-Variation Equations, we can prove that the left-hand side "output" of these bounds, satisfies the "input" bounds.

Therefore it is important to know whether we can restrict the domain of the $\{r_{ij}\}$, at which we have to determine bounds on $F(r_{12}, \ldots, r_{(n-1)n}, g^2, d)$ at intermediate stages of the calculation, to domains where inequalities such as the triangle inequalities, $r_{ij} + r_{jk} \geq r_{ik}$, are valid. These inequalities are valid for all known Euclidean spaces, namely those for positive integer d. There are further inequalities, valid for all known Euclidean spaces, and independent of the triangle inequalities, which state, for example, that the sum of the areas of three faces of a tetrahedron, is greater than or equal to the area of the fourth face. That these inequalities are independent of the triangle inequality, can be seen by considering an example such as

$$r_{12} = r_{13} = r_{23} = r$$

$$r_{14} = r_{24} = r_{34} = s \tag{8.1}$$

where

$$\frac{r}{2} < s < \frac{r}{\sqrt{3}} \tag{8.2}$$

These inequalities can be expressed in terms of the $\{r_{ij}\}$ by means of Heron's formula [64], that the area of a triangle whose sides have lengths a, b, and c, is

$$\frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a+c-b)(b+c-a)} \tag{8.3}$$

Heron's formula, and the higher-dimensional volumes, required to express the higher-dimensional analogues of the triangle and tetrahedron inequalities, all of which are valid for all known Euclidean spaces, can be expressed in terms of Gramm determinants: in r-dimensional Euclidean space, the square of the volume of an r-dimensional simplex, (generalized triangle or tetrahedron), with vertices at $x_1, \ldots, x_{r+1}$, is given by $\left(\frac{1}{r!}\right)^2$ times

$$\begin{vmatrix} (x_1 - x_{r+1})_1 & (x_1 - x_{r+1})_2 & \ldots & (x_1 - x_{r+1})_r \\ (x_2 - x_{r+1})_1 & (x_2 - x_{r+1})_2 & \ldots & (x_2 - x_{r+1})_r \\ \vdots & \vdots & \ddots & \vdots \\ (x_r - x_{r+1})_1 & (x_r - x_{r+1})_2 & \ldots & (x_r - x_{r+1})_r \end{vmatrix} \quad \times$$

$$\times \quad \begin{vmatrix} (x_1 - x_{r+1})_1 & (x_2 - x_{r+1})_1 & \dots & (x_r - x_{r+1})_1 \\ (x_1 - x_{r+1})_2 & (x_2 - x_{r+1})_2 & \dots & (x_r - x_{r+1})_2 \\ \vdots & \vdots & \ddots & \vdots \\ (x_1 - x_{r+1})_r & (x_2 - x_{r+1})_r & \dots & (x_r - x_{r+1})_r \end{vmatrix} \quad =$$

$$= \begin{vmatrix} (x_1 - x_{r+1})^2 & (x_1 - x_{r+1}).(x_2 - x_{r+1}) & \dots & (x_1 - x_{r+1}).(x_r - x_{r+1}) \\ (x_2 - x_{r+1}).(x_1 - x_{r+1}) & (x_2 - x_{r+1})^2 & \dots & (x_2 - x_{r+1}).(x_r - x_{r+1}) \\ \vdots & \vdots & \ddots & \vdots \\ (x_r - x_{r+1}).(x_1 - x_{r+1}) & (x_r - x_{r+1}).(x_2 - x_{r+1}) & \dots & (x_r - x_{r+1})^2 \end{vmatrix} \tag{8.4}$$

The dot products can then be expressed in terms of the $\{r_{ij}\}$ by means of

$$(x_i - x_{r+1}).(x_j - x_{r+1}) = \frac{1}{2}\left((x_i - x_{r+1})^2 + (x_j - x_{r+1})^2 - (x_i - x_j)^2\right) \quad =$$

$$= \quad \frac{1}{2}\left(r_{i(r+1)}^2 + r_{j(r+1)}^2 - r_{ij}^2\right) \tag{8.5}$$

The borderline case, where any of these inequalities becomes an equality, corresponds to the vanishing of the volume of the $(r+1)$-dimensional simplex, whose faces are the r-dimensional simplexes involved.

8.1.2 The need for a Gaussian representation of the area law

To perform the d-dimensional integrals, we can attempt to represent $F(r_{12}, \dots, r_{(n-1)n}, g^2, d)$ by a Laplace transform in the variables $\{r_{ij}^2\}$:

$$F(r_{12}, \dots, r_{(n-1)n}, g^2, d) \quad =$$

$$= \int_0^\infty \dots \int_0^\infty \mathrm{d}s_{12} \dots \mathrm{d}s_{(n-1)n} \tilde{F}(s_{12}, \dots, s_{(n-1)n}, g^2, d) e^{-(s_{12} r_{12}^2 + \dots + s_{(n-1)n} r_{(n-1)n}^2)} \tag{8.6}$$

The function $\tilde{F}(s_{12}, \dots, s_{(n-1)n}, g^2, d)$, if it exists, is formally given by Bromwich's integral over the $\{r_{ij}^2\}$, which is an integral over lines parallel to the imaginary axis, in the complex r_{ij}^2 planes, where we do not expect to have good information on $F(r_{12}, \dots, r_{(n-1)n}, g^2, d)$. Nevertheless, we can apply Bromwich's integral to seek a representation of $F(r_{12}, \dots, r_{(n-1)n}, g^2, d)$ in the form (8.6), provided the integrals converge. In practice, it might be important to know whether we can assume that the $\{r_{ij}\}$, in the d-dimensional Gaussian integrals, can be assumed to satisfy the triangle inequality, and its higher-dimensional generalizations, as discussed above, because if

so, we might be able to choose a form of $F(r_{12}, \ldots, r_{(n-1)n}, g^2, d)$, in the regions of real $\{r_{ij}^2\}$ where these inequalities are violated, which leads to a more convenient form of $\tilde{F}(s_{12}, \ldots, s_{(n-1)n}, g^2, d)$.

In practice, it is a non-trivial problem to find an explicit representation of the form (8.6) even for $e^{-\sigma A}$, where A is the area of the triangle with edge lengths r_{12}, r_{13}, and r_{23}, as given by Heron's formula, and we can see that it might be important to have the freedom to choose $F(r_{12}, r_{13}, r_{23}, g^2, d)$ to have a convenient dependence on the r_{ij}, for real $\{r_{ij}^2\}$ such that the triangle inequalities are violated. To find an explicit representation of the form (8.6) for $e^{-\sigma A}$, where A is the area of the minimal-area spanning surface of a loop formed of four straight segments, we might expect it to be important to have the freedom to choose a convenient form of $F(r_{12}, \ldots, r_{34}, g^2, d)$, for real $\{r_{ij}^2\}$ such that any of the triangle inequalities or tetrahedron inequalities are violated.

8.1.3 Application of Douglas's functional

Conceivably, representations in the form (8.6) of $e^{-\sigma A}$, where A is the area of the minimal-area spanning surface of a loop formed of n straight segments, might not actually be needed. As we discussed after equation (4.78), the area of the minimal-area orientable spanning surface of *any* simple closed path may be expressed in the quadratic form (4.77), summed over *all* d dimensions in which the path exists, provided the *parametrization* of the path is the parametrization for which (4.77) takes its minimum value [39], [65]. If we define $s = \tan\left(\frac{\theta}{2}\right)$, $t = \tan\left(\frac{\phi}{2}\right)$, (4.77), summed over all d dimensions in which the path exists, can be expressed in the form:

$$A \leq \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{(x(\theta) - x(\phi))^2}{4 \sin^2\left(\frac{\theta - \phi}{2}\right)} \mathrm{d}\theta \mathrm{d}\phi \tag{8.7}$$

where x is now a d-vector, and the inequality becomes an equality, for the parametrization of the path that minimizes the right-hand side.

We can explicitly introduce a reparametrization of the path into (8.7), by introducing a continuous real function f, such that $\theta = f(u)$, and $\phi = f(v)$, and $f(0) = 0$ and $f(2\pi) = 2\pi$. We may assume that f is monotonic, so that the reparametrization is invertible, since a parametrization, that involved doubling backwards and forwards along the path, would certainly not minimize the right-hand side of (8.7). The reparametrized

form of (8.7) is then given by:

$$A \leq \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{(x(u)-x(v)^2)}{4\sin^2\left(\frac{f(u)-f(v)}{2}\right)} \frac{\mathrm{d}f(u)}{\mathrm{d}u} \frac{\mathrm{d}f(v)}{\mathrm{d}v} \mathrm{d}u\mathrm{d}v \tag{8.8}$$

Let us now suppose that our simple closed path consists of n straight line segments, whose vertices are at $x_1, \ldots, x_n$. Then since, at any finite stage of our limiting procedure, discussed in connection with equation (5.54), the closed path $x_1, \ldots, x_n$, that defines our long-distance factor (5.38), only approximately follows the path around the edge of the relevant simply-connected window of a Group-Variation Equation diagram, it is natural to consider, in conjunction with the discretization of the path, a "discretization" of Douglas's variational problem. (The n in (5.38) has no connection with our present n, and is in fact equal to 1, since we are considering, at present, the vacuum expectation value of a single Wilson loop.) This means that, instead of seeking the minimum of the right-hand side of (8.8) among all continuous monotonic real functions f, such that $f(0) = 0$ and $f(2\pi) = 2\pi$, we will only seek its minimum among piecewise linear such f, which are completely defined, for all $0 \leq u \leq 2\pi$, by the $(n-1)$ values

$$f_r = f\left(\frac{2\pi r}{n}\right) \qquad\qquad 1 \leq r \leq (n-1) \tag{8.9}$$

such that

$$0 \equiv f_0 \leq f_1 \leq f_2 \leq \ldots \leq f_{n-1} \leq f_n \equiv 2\pi \tag{8.10}$$

If we suppose that our x_s, $1 \leq s \leq n$, are n samples of a d-vector function $x(u)$, it is clear, from the structure of (8.8), that we should consider the samples to be taken at $u = \frac{2\pi\left(s-\frac{1}{2}\right)}{n}$, $1 \leq s \leq n$, so that

$$x_s = x\left(\frac{2\pi\left(s-\frac{1}{2}\right)}{n}\right) \qquad\qquad 1 \leq s \leq n \tag{8.11}$$

Thus we define, as the discrete version of (8.8):

$$A_D\left(x_1, \ldots, x_n, f_1, \ldots, f_{n-1}\right) \equiv \frac{1}{4\pi} \sum_{\substack{1 \leq s \leq n \\ 1 \leq t \leq n \\ s \neq t}} \frac{\left(x_s - x_t\right)^2 \left(f_s - f_{s-1}\right)\left(f_t - f_{t-1}\right)}{4\sin^2\left(\frac{f_{s-1}+f_s-f_{t-1}-f_t}{4}\right)} \tag{8.12}$$

where the subscript D stands for Douglas, and we expect that minimizing $A_D\left(x_1, \ldots, x_n, f_1, \ldots, f_{n-1}\right)$ with respect to the f_s, $1 \leq s \leq (n-1)$, subject to (8.10),

will give an approximation to the area of the minimal-area spanning surface of the closed loop of straight line segments, with vertices $x_1, \ldots, x_n$.

We therefore might consider, as an approximation to $e^{-\sigma A}$, where A is the area of the actual minimal-area spanning surface of this closed loop of straight line segments, a constant, or slowly varying, multiple of the expression

$$\int_0^{2\pi} \mathrm{d}f_1 \ldots \int_0^{2\pi} \mathrm{d}f_{n-1} \theta(f_2 - f_1) \ldots \theta(f_{n-1} - f_{n-2}) e^{-\sigma A_D(x_1,\ldots,x_n,f_1,\ldots,f_{n-1})} \tag{8.13}$$

where $\theta(f)$ is the step function, as in (1.1), on the basis that a steepest descents approximation, to the integral (8.13), will give an approximation to $e^{-\sigma A}$, times some pre-exponential factor. What this pre-exponential factor will be, as well as the question of whether (8.13) can be generalized to higher-topology minimal-area spanning surface terms, such as the cylinder-topology term, in our ansatz for the long-distance behaviour of the correlation function of two Wilson loops, I leave to further work. If the pre-exponential factor turns out to have a non-trivial dependence on the x_s, $1 \leq s \leq n$, we could try to cancel it, at least in part, by including an additional function of the f_s, $1 \leq s \leq (n-1)$, in the integrand of (8.13), and then cancel any remaining dependence on the x_s by another factor represented in the form (8.6), provided a suitable $\tilde{F}$ exists.

The minimization of (8.12) with respect to the f_r, $1 \leq r \leq (n-1)$, subject to (8.10), is not expected to give a good representation of the areas of the minimal-area spanning surfaces, of general paths formed of n straight-line segments, with vertices at $x_1, \ldots, x_n$, because the piecewise-linear reparametrization of the path, defined by the f_r, does not have enough freedom. Rather, we would attempt to determine whether the representation (8.13) gives an adequate realization of the Wilson area law, when substituted into a path integral such as (1.27), where the kinematic factor, in (1.27), will suppress the contributions of paths where the successive x_s are distant from one another, rather than following, in some approximation, the straight line between the endpoints x and y of the path. For such paths, where the successive x_s follow, in some approximation, some smooth curve, and just add jagged "detail" to the curve, we might expect that the piecewise-linear reparametrization function, defined by the f_r, will now begin to have enough freedom to approach the parametrization, of the corresponding smooth curve, that minimizes Douglas's functional (8.8).

Indeed, if the curve $x(\theta)$, in (8.7), takes the form of a simple planar polygon, then the parametrization of the curve, for which (8.7) takes its minimum value, is the parametrization where θ represents the position on the unit circle in the complex plane,

while the perimeter of the polygon is swept out, as θ goes around the unit circle, by the Schwartz-Christoffel transformation.

Let the complex variable w, be defined as an analytic function of the complex variable z, by the equation:

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \kappa \left(e^{i\phi_1} - z\right)^{-\alpha_1} \left(e^{i\phi_2} - z\right)^{-\alpha_2} \ldots \left(e^{i\phi_n} - z\right)^{-\alpha_n} \tag{8.14}$$

where $\phi_1, \phi_2, \ldots, \phi_n$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are real angles, κ is a fixed complex number, and for definiteness, we assume that $0 < \phi_1 < \phi_2 < \ldots < \phi_n < 2\pi$. Then since the right-hand side of (8.14) is analytic inside, and on, the unit circle, in the z plane, except at the points on the unit circle where $z = e^{i\phi_s}$, $1 \leq s \leq n$, it follows that, if z traces out any simple closed curve, either in, or on, the unit circle, in the z-plane, avoiding the points $z = e^{i\phi_s}$, $1 \leq s \leq n$, on the unit circle, w will also trace out a closed curve in the w plane. In particular, on the unit circle, with $z = e^{i\theta}$, we see that

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}\theta} = iz\frac{\mathrm{d}w}{\mathrm{d}z} = i\kappa(2i)^{-(\alpha_1+\alpha_2+\ldots+\alpha_n)} e^{\frac{-i}{2}(\phi_1\alpha_1+\phi_2\alpha_2+\ldots+\phi_n\alpha_n)} e^{\frac{i\theta}{2}(2-(\alpha_1+\alpha_2+\ldots+\alpha_n))} \quad \times \\ \times \quad \sin^{-\alpha_1}\left(\frac{\phi_1-\theta}{2}\right)\sin^{-\alpha_2}\left(\frac{\phi_2-\theta}{2}\right)\ldots\sin^{-\alpha_n}\left(\frac{\phi_n-\theta}{2}\right)\end{aligned} \tag{8.15}$$

Thus if

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n = 2 \tag{8.16}$$

the phase of $\frac{\mathrm{d}w}{\mathrm{d}\theta}$ is constant, while θ is strictly between any two consecutive ϕ_s. Let us now assume (8.16), and also that all the α_s are strictly less than 1, so that (8.14) is integrable along the unit circle. We see that if z follows a contour that proceeds anticlockwise around the unit circle, except that, to avoid the points $e^{i\phi_s}$, $1 \leq s \leq n$, it deviates inside the unit circle, along semicircles of very small radius, ϵ, centred at these points, the path followed in the w plane, in the limit where $\epsilon \to 0$, has the form of a polygon, where the phase of $\frac{\mathrm{d}w}{\mathrm{d}\theta}$ increases by $\pi\alpha_s$, as θ passes α_s.

Furthermore, since

$$\left|\frac{\mathrm{d}w}{\mathrm{d}\theta}\right| = \frac{|\kappa|}{4}\left|\sin\left(\frac{\phi_1-\theta}{2}\right)\right|^{-\alpha_1}\left|\sin\left(\frac{\phi_2-\theta}{2}\right)\right|^{-\alpha_2}\ldots\left|\sin\left(\frac{\phi_n-\theta}{2}\right)\right|^{-\alpha_n} \tag{8.17}$$

we see that, in the region of a convex angle of the polygon, or in other words, an angle $\pi\alpha_s$ such that $\alpha_s > 0$, w sweeps very rapidly through the region of the corner, and infinitely rapidly through the corner itself, as θ passes steadily through θ_s.

This means that the parametrization of a polygonal curve, which minimizes Douglas's functional (8.7), must be such that, near any convex angle of the polygon, an

increasingly large length of the perimeter of the polygon must be swept out, for a fixed increase of the parameter along the path, as the corner is approached. Or in other words, the derivative of the parameter, with respect to distance along the perimeter of the polygon, must tend to zero, as a convex corner of the polygon is approached.

In particular, if the path $x(u)$ in (8.8), is a simple planar polygon, and the parameter u is the distance along the perimeter of the polygon, then the function $f(u)$, in (8.8), which minimizes (8.8), will be the angle θ, in the Schwartz-Christoffel transformation, and $\frac{\mathrm{d}f(u)}{\mathrm{d}u}$ will be a constant multiple of $\frac{1}{\left|\frac{\mathrm{d}w}{\mathrm{d}\theta}\right|}$.

Thus the monotonic function $f(u)$, such that $f(0) = 0$, and $f(2\pi) = 2\pi$, which minimizes (8.8), will be such that $\frac{\mathrm{d}f(u)}{\mathrm{d}u}$ tends to zero as $|u-u_0|^{\left(\frac{\alpha}{1-\alpha}\right)}$, as u approaches a point, u_0, where the curve turns through a convex angle $\pi\alpha$, if the curve $x(u)$ is a simple planar polygon, and u is proportional to distance along the curve. In particular, if the curve is a rectangle, then $\frac{\mathrm{d}f(u)}{\mathrm{d}u}$ tends to zero, linearly, at each corner. The approximate representation of the minimization of (8.8), as the minimum of (8.12), with respect to the f_r, $1 \le r \le (n-1)$, subject to (8.10), simply doesn't have enough flexibility to cope with this, if n is small, and the points x_s don't approximately follow some smooth path.

To see what this means in an explicit example, let us suppose that $n = 4$, and the points x_s, $1 \le s \le 4$, form a rectangle, with sides of lengths A and B. Since the $f_s - f_{s-1}$ are associated with the corners of the rectangle, we assume they are all equal to $\frac{\pi}{2}$. Then (8.12) gives the value $\frac{3\pi}{16}(A^2+B^2) \ge \frac{3\pi}{8}AB = 1.178\ AB$ for the area of the rectangle.

We might suppose that the situation improves if we allow more vertices along the perimeter of the rectangle, but instead, a new problem arises. Let us now suppose that $n = 8$, and we have the same rectangle as before, with a vertex at each corner, and a vertex at the middle of each side. Then, by the symmetries of the rectangle, we assume there can be three possible values of $f_s - f_{s-1}$, namely a value a, for s in the middle of a side of length A, a value b, for s in the middle of a side of length B, and a value c, for s at a corner, such that $2a + 2b + 4c = 2\pi$. We find that the $\sin^2\left(\frac{f_{s-1}+f_s-f_{t-1}-f_t}{4}\right)$, in the denominators in (8.12), depend on a, b, and c, only through the combination $a-b$, and some of them are independent of a, b, and c. If $A = B$, we assume $a = b$, by symmetry, and we then find that (8.12) takes its minimum value at $c = 0$, which is on the boundary of the region allowed by (8.10). Since c is the value of $f_s - f_{s-1}$, for s at a corner, this is not unexpected, in view of the discussion of the Schwartz-Christoffel

transformation. For general A and B, and general a and b, subject to $c = 0$, so that $a + b = \pi$, we find that (8.12) gives:

$$\frac{1}{8\pi}\left(2(A^2 + B^2)ab + A^2b^2 + B^2a^2\right) \tag{8.18}$$

This attains its minimum value, in the region allowed by (8.10), either at $a = \pi$, $b = 0$, or $a = 0$, $b = \pi$, and the minimum value it takes is

$$\frac{\pi}{8}\min(A^2, B^2) \tag{8.19}$$

which is *less* than the correct value, AB. However, we see that, when $c = 0$, and either a or b is also equal to 0, some of the $\sin^2\left(\frac{f_{s-1}+f_s-f_{t-1}-f_t}{4}\right)$, in the denominators in (8.12), vanish, although none of them vanish for $c = 0$, if both a and b are non-zero. In particular, suppose that $A > B$, so that the minimum of (8.18) is attained at $a = \pi$, $b = 0$. The terms in (8.12), which come from s and t at the ends of an edge of the square, of length B, are:

$$\frac{B^2c^2}{4\pi\sin^2\left(\frac{b+c}{2}\right)} \tag{8.20}$$

for general a, b, and c. We see that, although this term vanishes when the point $a = \pi$, $b = c = 0$ is approached by the route $c \to 0$ for nonzero b, followed by $b \to 0$, it can attain any value from 0 to $\frac{B^2}{\pi}$, by approaching the point $a = \pi$, $b = c = 0$, keeping different fixed values of the ratio b/c. Thus (8.12) is not well-behaved, near the boundary of the allowed domain, (8.10).

In fact, even if we take the value $a = b = \frac{\pi}{2}$ in (8.18), we get the value $\frac{3\pi}{32}(A^2 + B^2)$, which for $A = B$ is *less* than the correct value, AB. Thus it appears that the minimum value of (8.12) can be less than the correct value of the area of the minimal-area spanning surface, of the path of straight line segments, with ends at the x_s, if the f_r are allowed to approach the boundaries of the allowed region, (8.10). This problem must be due to the approximate way in which (8.12) represents the problem of minimizing (8.8), in the restricted space of piecewise-linear $f(u)$, with the points at which $\frac{\mathrm{d}f(u)}{\mathrm{d}u}$ is allowed to change, being halfway between the values of u at successive corners of the path of straight segments, since (8.8), correctly calculated, is strictly bounded below, by the true area of the minimal-area spanning surface of the path of straight segments, and attains this value, only for $f(u)$ that solve the variational problem. (The $f(u)$ that solve the variational problem are not unique, due to the freedom to re-parametrize the unit circle, by conformal transformations.) Thus we might attempt to solve this

problem by replacing (8.12) by a more accurate representation of (8.8), calculated with piecewise-linear $f(u)$, or by finding an appropriate subdomain of the allowed domain (8.10), and a corresponding subdomain of the integration domain in (8.13).

We observe that the requirement, noted above, that for an $f(u)$ that minimizes (8.8), $\frac{\mathrm{d}f(u)}{\mathrm{d}u}$ must tend to zero, as u approaches a "convex" corner of the path, if u represents the distance along the path, can be re-expressed, for the approximate discrete form (8.12), by saying that if, in (8.12), we want the minimum to be attained for approximately constant values of the f_r, then the "smooth" path, which the vertices approximately follow, must be more sparsely populated by vertices, where it has a "convex" bend, or in other words, where it deviates maximally from its "main" route, and more densely populated by vertices, where it is approximately following its "main" route. This will automatically be the case, for typical paths, in path integrals such as (1.27).

In the search for a generalization of the representation (8.13), for the correlation function of two or more Wilson loops, we might consider the way in which (8.13), which is based on a discretization of the minimisation problem of Douglas's functional, nevertheless has some resemblance, in the integration domain of the f_r, $1 \leq r \leq (n-1)$, to a dual resonance amplitude.

8.1.4 The non-island diagrams spoil the Gaussian representation

When we calculate a non-island diagram in the right-hand sides of the Group-Variation Equations, the Gaussian representation (8.6) will be spoilt, when we integrate one end of a 45-path along a straight-line segment of one of the left-hand side Wilson loops. This will give an incomplete Gaussian integral, which we then have to re-represent in the form (8.6), with a suitable new $\tilde{F}$, if further d-dimensional integrals have to be performed.

8.1.5 The short-distance factors

The short-distance factors, which we divide the vacuum expectation values and correlation functions by, to obtain the long-distance factors (5.38), can be chosen in any convenient form, provided they cancel the linear divergences along the Wilson loops, and can be calculated in perturbation theory. They should also soften the dependence

of the long-distance factors on the fine details of the paths, which would be expected to occur, for example, for the form analysed in connection with equations (5.38) to (5.53).

Since the short-distance factors are going to be restored in the windows of the right-hand side Group-Variation equation diagrams, by means of effective fields, as we did in the simplest case, in the calculation from equation (6.15) to (6.40), they can in principle be defined by effective field theories, that depend on the shapes of the individual Wilson loops. This means that there might be a possibility of defining gauge-invariant short distance factors, by the use of gauge-field actions of the form:

$$\frac{1}{4}\int \mathrm{d}^d x \frac{1}{g^2(x)} F_{\mu\nu a} F_{\mu\nu a} \tag{8.21}$$

where $g^2(x)$ is a function of the shortest distance from x to any point on the loop, and is chosen to equal the same value of g^2 as used in the Wilson loop vacuum expectation values and correlation functions, when this distance is zero, and to decrease sufficiently rapidly, as this distance increases, that these short-distance factors can be calculated in perturbation theory. The use of a gauge-field action such as (8.21) means that the free propagator used in the short-distance factor will not be translation-invariant, and I leave it to future work to determine whether such a technique is practical.

Given any two distinct choices of short-distance factor, their ratio should be calculable in perturbation theory, and we should then find, that if we multiply a long-distance factor, calculated with one particular choice of short-distance factor, by the appropriate ratio of short-distance factors, we should obtain the long-distance factor, as calculated with the other short-distance factor, which might provide a useful check on the calculations.

8.1.6 Residual logarithmic divergences of subdiagrams with two "path legs" and one gluon leg

A final point to be resolved, in the definition of the long-distance factors, (5.38), is the treatment of the residual logarithmic divergences of subdiagrams that have two "legs" that form part of the Wilson loop, and one gluon leg, as discussed after equation (5.53). Within dimensional regularization, these divergences might be removed by the renormalization of the gluon field $A_{\mu a}^{(x)}$ at the vertex, as was discussed by Dotsenko and Vergeles [43].

8.1.7 The need to test the prescriptions by going backwards and forwards between the Group-Variation Equations and renormalization-group-improved perturbation theory at short distances

All the prescriptions will then need to be tested by going backwards and forwards between the Group-Variation Equations and renormalization-group-improved perturbation theory, at short distances, or in other words, for a small value of the input coupling constant g^2, using the limiting procedure for calculating the path integrals, as described after equation (5.54).

8.1.8 $F_{\mu\nu}$ insertions in the sums over paths

One final prescription needed, is for dealing with the $\bar{F}_{\mu\nu}$ insertions in the path-ordered phase factors, that occur in the sums over paths for the gluon propagator, as in equation (1.18).

The solution to equation (1.15) can be expressed slightly more neatly if we define $H_{\sigma\nu}$ by:

$$\left(-\bar{D}^2\delta_{\mu\sigma} - 2\bar{F}_{\mu\sigma}\right) H_{\sigma\nu} = \delta_{\mu\nu} \tag{8.22}$$

We then define

$$\tilde{E} \equiv -\bar{D}_\mu H_{\mu\nu} \bar{D}_\nu \tag{8.23}$$

The solution to (1.15) can then be written as:

$$g^2 \left(\begin{array}{c|c} H_{\sigma\nu} + H_{\sigma\tau}\bar{D}_\tau \left(\frac{1-\beta}{(1-\beta)\tilde{E}+\beta}\right) \bar{D}_\alpha H_{\alpha\nu} & H_{\sigma\tau}\bar{D}_\tau \left(\frac{i}{(1-\beta)\tilde{E}+\beta}\right) \\ \left(\frac{-i}{(1-\beta)\tilde{E}+\beta}\right) \bar{D}_\alpha H_{\alpha\nu} & \left(\frac{1-\tilde{E}}{(1-\beta)\tilde{E}+\beta}\right) \end{array} \right) \tag{8.24}$$

All the $\bar{F}_{\sigma\tau}$ insertions are now contained within $H_{\mu\nu}$. Furthermore, $\tilde{E}_{Ax,By}$ is equal to $\delta_{AB}\delta^4(x-y)$, plus terms of degree two and higher in $A_{\mu a}$, so the expansion

$$\frac{1}{\tilde{E}} = \frac{1}{1-(1-\tilde{E})} = 1 + (1-\tilde{E}) + (1-\tilde{E})^2 + \ldots \tag{8.25}$$

is slightly improved, in comparison with equation (1.30).

We can then express the $\bar{F}_{\mu\nu}$ insertions, in $H_{\alpha\beta}$, by using that, by equation (1.29):

$$\left(\bar{D}_\mu \bar{D}_\nu \left(\frac{-1}{\bar{D}^2}\right)\right)_{Ax,By} \quad \simeq$$

$$\simeq \int \mathrm{d}^4 w \frac{(w-x)_\mu}{2\sigma} \frac{e^{-\frac{(w-x)^2}{4\sigma}}}{(4\pi\sigma)^2} W_{Ax,Ew} \int \mathrm{d}^4 z \frac{(z-w)_\nu}{2\sigma} \frac{e^{-\frac{(z-w)^2}{4\sigma}}}{(4\pi\sigma)^2} W_{Ew,Cz} \left(\frac{-1}{\bar{D}^2}\right)_{Cz,By} \tag{8.26}$$

We thus see that an $F_{\mu\nu}$ insertion between two expressions (1.27), corresponds to inserting two extra straight segments, with the usual segment weights as in (1.27), and an additional pre-exponential factor:

$$\frac{(w-x)_\mu}{2\sigma} \frac{(z-w)_\nu}{2\sigma} - \frac{(w-x)_\nu}{2\sigma} \frac{(z-w)_\mu}{2\sigma} \tag{8.27}$$

As noted in connection with equations (6.15) to (6.40), it is these $\bar{F}_{\mu\nu}$ insertions that are responsible for reversing the sign of the island diagram contributions.

8.1.9 The question of whether there might be any further independent relations involving the distances among a finite set of points, for general complex d

It is perhaps worth asking, whether there are any further restrictions, in addition to the triangle inequalities, and their higher-dimensional generalizations, as discussed above, that might be imposed on real $\{r_{ij}\}$, all ≥ 0, in a d-dimensional Euclidean space, for general complex d. For example, for integer n, $n \geq 3$, if an arbitrary set of real $\{r_{ij}\}$, $1 \leq i < j \leq n$, all ≥ 0, is given, which satisfy all the triangle inequalities, and their higher-dimensional generalizations, up to, and including, the $(n-1)$-dimensional simplex inequalities, can a set of n points always be found, in some d-dimensional Euclidean space, with integer $d \geq 0$, that realizes this set of $\{r_{ij}\}$?

8.2 BPHZ Renormalization, Pauli-Villars Regulators and Higher Derivative Terms, and Lattice Regularization

I have discussed BPHZ renormalization [66], [67], [68], [69], [26], in connection with the renormalization group, (equations (5.27) to (5.37), and with the reversal of the sign of the island diagram contributions, (equations (6.15) to (6.40)).

In [26], in an effort to understand better how renormalization works in position space, I have given a BPHZ convergence proof directly in position space, without any parametrizations of the propagators, by finding a method of cutting up the Cartesian

product of the configuration space of the vertices, and the set of all the forests, into a finite number of sectors, each of which is the Cartesian product of a subset of the configuration space of the vertices, and a set of forests, called a "good set of forests," such that the position space integrals can be adequately bounded, and proved to converge, in each such sector. I have given a brief discussion of this method in connection with equations (6.6) to (6.10).

The outstanding problem is to find the additional finite counterterms that restore the Ward-Takahashi identities and the Slavnov-Taylor identities. A simple example of the restoration of a Ward-Takahashi identity is given in equation (5.12). Recent work by Grassi, Hurth, and Steinhauser, [70], might be helpful in the search for further finite counterterms.

The method of higher-derivative terms in the action, plus Pauli-Villars fields to regulate the one-loop divergences, [71], [21], [72], is natural for proving the absence of anomalies in higher-loop orders, for example, the Adler-Bardeen theorem [73], because the higher-derivative terms covariantly regularize the higher-loop divergences, but not the one-loop divergences. Lee and Zinn-Justin, [21], used scalar and spinor Pauli-Villars fields to regularize the one-loop divergences, while Slavnov, [71], used vector Pauli-Villars fields. Some difficulties, which arose in the method with vector Pauli-Villars fields, are discussed, with a possible resolution, in reference [72].

In connection with the Group-Variation Equations, it is interesting to note that, due to the use of Landau gauge, it is sufficient to include higher-derivative terms in the gauge-invariant part of the action density, $\frac{1}{4g^2}F_{\mu\nu a}F_{\mu\nu a}$. The gauge-fixing and Fadeev-Popov terms can be used exactly in the forms (1.6), and (1.7), at least, at the level of ordinary perturbation theory. (The analogue of equations (1.16) to (1.18), or of (8.22) to (8.24), in the presence of higher-derivative terms in the $\frac{1}{4g^2}F_{\mu\nu a}F_{\mu\nu a}$ part of the action density, has not yet been worked out.) For non-zero α and β in equation (1.6), the free gluon propagator has a term that is not properly regularized, if we only include higher-derivative terms in the $\frac{1}{4g^2}F_{\mu\nu a}F_{\mu\nu a}$ part of the action density, but for $\alpha = \beta = 0$, or in other words, for Landau gauge, this term vanishes. Power-counting arguments also indicate that, for Landau gauge, there is no need for any higher-derivative terms in the gauge-fixing, Fadeev-Popov, or Pauli-Villars actions, in order to obtain a negative degree of divergence, for all subdiagrams with two or more loops.

It would seem possible, furthermore, that the gauge-invariant higher-derivative terms in $\frac{1}{4g^2}F_{\mu\nu a}F_{\mu\nu a}$ will lead, in Landau gauge, to regularization of the linear di-

vergences along Wilson loops, and softening their dependence on the fine details of the path, so that it would not be necessary to divide by short-distance factors. There will be higher-dimension gauge-invariant insertions along Wilson loops, in the analogues of the propagators (1.16) to (1.18), or (8.22) to (8.24), in the presence of background fields in the subgroup, but these would be expected to be expressible in terms of extra segments along the path, with suitable extra pre-exponential weight factors, analogously to (1.29) and (8.26). If this possibility is utilized, it will be necessary to ensure that the correct dependence, on the fine details of the paths, which would normally be taken care of by the short-distance factors, is included in the ansatz for the vacuum expectation values and correlation functions.

An extra complication which we might face in this approach, in addition to the extra vertices coming from the higher covariant derivative terms in the gauge-invariant part of the gauge-field action, would be due to the possibility that we might have to allow the Pauli-Villars regulator fields to wander in and out of the subgroup. This would mean that the analogues of the propagators (1.16) to (1.18), or (8.22) to (8.24), would have to be generalized to higher-dimensional propagator matrices, with additional rows and columns for the Pauli-Villars regulator fields, which would have to be calculated in the presence of both background gauge fields in the subgroup, and background Pauli-Villars regulator fields in the subgroup.

A compromise might involve using higher covariant derivative terms, in the Yang-Mills action, to regularize the overall divergences of subdiagrams with two or more loops, and BPHZ counterterms, with appropriate finite counterterms to restore the Slavnov-Taylor and Ward-Takahashi identities, for the one-loop divergences.

With regard to lattice regularization, I do not know, at present, whether a gauge-fixing such as the Landau-gauge case of (1.6) and (1.7) is possible, which ensures that Fadeev-Popov loops stay either in the subgroup, or out of it.

In this connection, it is natural to ask, both in the continuum, and on the lattice, whether the Group-Variation Equations might be generalized to gauges other than Landau gauge, by defining suitable BRS-invariant generalizations of Wilson loops, that have the form of Wilson loops, with Fadeev-Popov field insertions. One would attempt to solve a generalization of equation (1.14), or (1.15), for 4 by 4 block matrices, that include rows and columns for the Fadeev-Popov fields not in the subgroup, and where the background fields now include the Fadeev-Popov fields in the subgroup, in addition to the gauge fields in the subgroup. Then one would attempt to express the propagators,

in the presence of the background fields in the subgroup, in terms of "core" non-local parts, that would be generalizations of $\left(\frac{-1}{D^2}\right)$, and could be represented as path integrals, that would be generalizations of the expression (1.27). Then one would attempt to extract the appropriate BRS-invariant generalizations of Wilson loops, which would become the physical observables, in gauges other than Landau gauge, analogous to the Wilson loops, in Landau gauge, by examination of the expressions for the generalized path integrals, analogous to (1.27). This, also, remains a topic for future work.

The renormalization of Wilson loops has been studied by Brandt, Neri, and Sato [74].

8.3 Derivation of the perturbative expansion of the propagator from the path integral form

Finally, I shall briefly sketch the derivation of equation (1.28), from the expression (1.27), as an additional check that $\left(\frac{-1}{D^2}\right)_{Ax,By}$ really can be represented as the $\sigma \to 0$ limit of the expression (1.27).

We first check that (1.27), with the path-ordered phase factor removed, is equal to the free, massless, scalar propagator, and indeed, using the four-dimensional version of (4.79), we find that (1.27) then becomes equal to:

$$\sigma \sum_{n=0}^{\infty} \frac{e^{-\frac{(x-y)^2}{4\sigma(n+1)}}}{(4\pi\sigma(n+1))^2} \tag{8.28}$$

This, in turn, becomes equal, in the limit $\sigma \to 0$, to:

$$\int_0^{\infty} \mathrm{d}s \frac{e^{-\frac{(x-y)^2}{4s}}}{(4\pi s)^2} = \frac{1}{4\pi^2(x-y)^2} = \left(\frac{-1}{\partial^2}\right)_{xy} \tag{8.29}$$

since the integrand in the left-hand side of (8.29) is Riemann-integrable from 0 to ∞, and (8.28) is a permitted approximation to the integral, within the definition of Riemann integrability, resulting from dividing the integration domain into intervals of size σ. For small, non-zero σ, the difference, between (8.28) and (8.29), is of order σ.

We now substitute equation (1.1) into equation (1.27), and seek the coefficient of

$$A^{(x_1)}_{\mu_1 a_1} A^{(x_2)}_{\mu_2 a_2} \ldots A^{(x_m)}_{\mu_m a_m} \left(t_{a_1} t_{a_2} \ldots t_{a_m}\right)_{AB} \tag{8.30}$$

In consequence of the path ordering, we may split the sequence of $A^{(x)}_{\mu a}$ into consecutive groups, which correspond to $A^{(x)}_{\mu a}$'s that are on the same straight segment in

the path-ordered phase factor $(W_{xz_1z_2\ldots z_ny})_{AB}$. Suppose that among the subindices, 1 to m, those that represent the first $A^{(x)}_{\mu a}$ in a group, or in other words, the first $A^{(x)}_{\mu a}$ on a new straight segment, are

$$1 = r_1 < r_2 < \ldots < r_p \leq m \tag{8.31}$$

Let us consider a fixed n in (1.27), (noting that $(n+1) \geq p$, since we need at least p distinct straight segments, for the term under consideration), and consider the contribution where, for $1 \leq q \leq p$, the q^{th} group of $A^{(x)}_{\mu a}$'s is on the segment, in (1.27), from z_{t_q} to z_{t_q+1}, such that

$$0 \equiv t_0 \leq t_1 < t_2 < \ldots < t_p < t_{p+1} \equiv (n+1) \tag{8.32}$$

and I have defined $z_0 \equiv x$, and $z_{n+1} \equiv y$, in the notation of (1.27).

We Taylor-expand all the $A^{(x)}_{\mu a}$ on a common segment, about the first end of that segment, in the sequence $z_0, z_1, z_2, \ldots, z_n, z_{n+1}$. To extract the coefficient of the contribution with one, or more, $A^{(x)}_{\mu a}$, or its derivatives, at each of these points, we set

$$z_{t_q} = w_q = \text{fixed} \qquad\qquad 1 \leq q \leq p \tag{8.33}$$

and integrate only over the other z_u's in (1.27), $1 \leq u \leq n$.

For the integral with fixed endpoints $z_0 = x$ and $z_{t_1} = w_1$, we get, if $0 < t_1$, the four-dimensional version of (4.79), with the $n+1$ in (4.79) set equal to $(t_1 - t_0) = t_1$, and the y in (4.79) set equal to w_1, while if $0 = t_1$, we simply get $\delta^4(x - w_1)$.

For the integral with fixed endpoints $z_{t_q} = w_q$ and $z_{t_{(q+1)}} = w_{q+1}$, $1 \leq q \leq p$, (with $w_{p+1} \equiv z_{t_{(p+1)}} = z_{n+1} = y$), we consider specific terms, in the Taylor expansions about w_q, of the $A^{(x)}_{\mu a}$'s on the straight segment from w_q to z_{t_q+1}, and do the path-ordered integrals along the segment, as in (1.1). The result is a numerical coefficient, times an integral, which is the four-dimensional version of the left-hand side of (4.79), with the $n+1$ in (4.79) set equal to $t_{q+1} - t_q$, the x in (4.79) set equal to w_q, the y in (4.79) set equal to w_{q+1}, and the integrand multiplied by a factor $(z_1 - w_q)_\mu$ for each $A^{(x)}_{\mu a}$ on the segment from w_q to z_{t_q+1}, and a factor $(z_1 - w_q)_\alpha$, for each derivative $(\hat{w}_q)_\alpha \equiv \frac{\partial}{\partial (w_q)_\alpha}$ that acts on a $A^{(w_q)}_{\mu a}$, in the Taylor expansion of a $A^{(x)}_{\mu a}$ on that segment, about $A^{(w_q)}_{\mu a}$.

The required integrals can all be obtained by suitable linear combinations of derivatives, with respect to x, of the four-dimensional version of (4.79). In particular, (denoting the derivative, with respect to a quantity, by that quantity with a hat over

it):

$$\int \mathrm{d}^4 z_1 \ldots \int \mathrm{d}^4 z_n (z_1 - x)_\alpha \frac{e^{-\frac{(x-z_1)^2}{4\sigma}}}{(4\pi\sigma)^2} \frac{e^{-\frac{(z_1-z_2)^2}{4\sigma}}}{(4\pi\sigma)^2} \ldots \frac{e^{-\frac{(z_n-y)^2}{4\sigma}}}{(4\pi\sigma)^2} = 2\sigma \hat{x}_\alpha \frac{e^{-\frac{(x-y)^2}{4\sigma(n+1)}}}{(4\pi\sigma(n+1))^2} \tag{8.34}$$

$$\int \mathrm{d}^4 z_1 \ldots \int \mathrm{d}^4 z_n (z_1 - x)_\alpha (z_1 - x)_\beta \frac{e^{-\frac{(x-z_1)^2}{4\sigma}}}{(4\pi\sigma)^2} \frac{e^{-\frac{(z_1-z_2)^2}{4\sigma}}}{(4\pi\sigma)^2} \ldots \frac{e^{-\frac{(z_n-y)^2}{4\sigma}}}{(4\pi\sigma)^2} =$$

$$= \left(2\sigma\delta_{\alpha\beta} + (2\sigma)^2 \hat{x}_\alpha \hat{x}_\beta\right) \frac{e^{-\frac{(x-y)^2}{4\sigma(n+1)}}}{(4\pi\sigma(n+1))^2} \tag{8.35}$$

$$\int \mathrm{d}^4 z_1 \ldots \int \mathrm{d}^4 z_n (z_1 - x)_\alpha (z_1 - x)_\beta (z_1 - x)_\gamma \frac{e^{-\frac{(x-z_1)^2}{4\sigma}}}{(4\pi\sigma)^2} \frac{e^{-\frac{(z_1-z_2)^2}{4\sigma}}}{(4\pi\sigma)^2} \ldots \frac{e^{-\frac{(z_n-y)^2}{4\sigma}}}{(4\pi\sigma)^2} =$$

$$= \left((2\sigma)^2 \left(\delta_{\alpha\beta}\hat{x}_\gamma + \delta_{\alpha\gamma}\hat{x}_\beta + \delta_{\beta\gamma}\hat{x}_\alpha\right) + (2\sigma)^3 \hat{x}_\alpha \hat{x}_\beta \hat{x}_\gamma\right) \frac{e^{-\frac{(x-y)^2}{4\sigma(n+1)}}}{(4\pi\sigma(n+1))^2} \tag{8.36}$$

We can show by induction that, for integer $a \geq 1$, if there are $(2a-1)$ or $(2a)$ factors, like $(z_1 - x)_\alpha$, in the left-hand side, of one of these equations, then the leading term, in the right-hand side, is of order σ^a.

We substitute these formulae into our contribution, specified by (8.31), (8.32), (8.33), and specific terms in the Taylor expansions of the $A^{(x)}_{\mu a}$, about the initial points of the straight segments they are on, or in other words, about the appropriate points w_q, $1 \leq q \leq p$, and we see that the result is that, firstly, for the integrals over the z_u, $1 \leq u < t_1$, we get a factor

$$\frac{e^{-\frac{(x-w_1)^2}{4\sigma t_1}}}{(4\pi\sigma t_1)^2} \tag{8.37}$$

if $0 < t_1$, and a factor $\delta^4(x - w_1)$ if $0 = t_1$.

And secondly, for each q, $1 \leq q \leq p$, and the integrals over the z_u, $t_q < u < t_{q+1}$, we get the appropriate derivatives acting on the $A^{(w_q)}_{\mu_v a_v}$, $r_q \leq v < r_{q+1}$, corresponding to the specific Taylor terms we are considering, times a numerical coefficient, that comes from the Taylor expansion coefficients, and the path-ordered integrals of the appropriate Taylor expansion factors, such as $((w_q)_\alpha + s_v(z_{t_q+1} - w_q)_\alpha)$, along the straight segment from w_q to z_{t_q+1}, times a factor that is the right-hand side of one of the series of equations that begins with (8.34) to (8.36), with x replaced by w_q, y replaced by w_{q+1}, and $(n+1)$ replaced by $(t_{q+1} - t_q)$.

We now sum over n in (1.27), (noting that the term we are considering, only arises for $(n+1) \geq p$), and over the t_q, $1 \leq q \leq p$, subject to (8.32). We define $b_q \equiv (t_{q+1} - t_q)$,

$0 \le q \le p$, and note that these sums are equivalent to summing over b_0 from 0 to ∞, and over the b_q from 1 to ∞, for $1 \le q \le p$. Then since the only dependence on b_q, for each $0 \le q \le p$, is through the final factor, in the right-hand side, of the appropriate equation, in the series of equations that begins with the four-dimensional version of (4.79), (8.34), (8.35), and (8.36), (except for the case $b_0 = 0$, which gives a factor $\delta^4(x - w_1)$), we see that this factor gets transformed, for each q, $0 \le q \le p$, into the expression (8.28), with x replaced by w_q, y replaced by w_{q+1}, and $(n+1)$ replaced by b_q, and the overall factor of σ removed, except that for $q = 0$, we get an additional term $\delta^4(w_0 - w_1)$, where I define w_0 to be equal to the x in (1.27).

We next note that for each q, $1 \le q \le p$, the right-hand side of the appropriate equation, in the series of equations that begins with (8.34) to (8.36), includes *at least* one factor of σ. Hence, with the overall factor of σ in (1.27), we have *at least* $(p+1)$ powers of σ, and we can assign one of these powers of σ to each of the $(p+1)$ versions of the expression (8.28), with the overall factor of σ removed, that we have obtained, (one for each q, $0 \le q \le p$). We see that we may therefore take the limit $\sigma \to 0$, and that the extra term $\delta^4(w_0 - w_1)$ in the sum, in the version of (8.28) we obtain for $q = 0$, is proportional to σ, and may be dropped, and that all terms of order σ^2, or higher, in the series of equations that begins with (8.34) to (8.36), may be dropped.

Thus for each q, $0 \le q \le p$, the final factor, in the right-hand side of the appropriate equation, in the series of equations that begins with the four-dimensional version of (4.79), (8.34), (8.35), and (8.36), has now been transformed to $\left(\frac{-1}{\partial^2}\right)_{w_q w_{q+1}}$. And furthermore, we get no contribution, unless, for all $1 \le q \le p$, $(r_{q+1} - r_q) \le 2$ holds, (where I define $r_{p+1} \equiv (m+1)$), and we have only the zeroth order Taylor terms, if $(r_{q+1} - r_q) = 2$, and only the zeroth order, or the first order, Taylor term, if $(r_{q+1} - r_q) = 1$. (More carefully, we would have to replace the Taylor expansions, by the sum of the terms we will keep, plus Taylor remainder terms, and bound and drop the contribution of every term that contains a Taylor remainder term.)

If $(r_{q+1} - r_q) = 1$, the zeroth-order Taylor term gives

$$2\, A^{(w_q)}_{\mu_{r_q} a_{r_q}} (\hat{w}_q)_{\mu_{r_q}} \left(\frac{-1}{\partial^2}\right)_{w_q w_{q+1}} \tag{8.38}$$

where the factor of 2 comes from the factor of 2 in the right-hand side of (8.35).

The first-order Taylor term gives

$$\frac{1}{2}\, 2\, \delta_{\mu_{r_q}\alpha} \left((\hat{w}_q)_\alpha A^{(w_q)}_{\mu_{r_q} a_{r_q}}\right) \left(\frac{-1}{\partial^2}\right)_{w_q w_{q+1}} \tag{8.39}$$

where the factor of $\frac{1}{2}$ comes from the integral of $((w_q)_\alpha + s_{r_q}(z_{t_q+1} - w_q)_\alpha)$ with respect to s_{r_q}, and the factor of 2 comes from the factor of 2, in the first term, in the right-hand side of (8.36).

If $(r_{q+1} - r_q) = 2$, the zeroth-order Taylor terms give

$$\frac{1}{2}\ 2\ \delta_{\mu_{r_q}\mu_{r_q+1}} A^{(w_q)}_{\mu_{r_q}a_{r_q}} A^{(w_q)}_{\mu_{r_q+1}a_{r_q+1}} \left(\frac{-1}{\partial^2}\right)_{w_q w_{q+1}} \tag{8.40}$$

where the factor of $\frac{1}{2}$ comes from the path-ordering, on the straight segment, from w_q to z_{t_q+1}, and the factor of 2 comes from the factor of 2, in the first term, in the right-hand side of (8.36).

The sum of (8.38), (8.39), and (8.40) is

$$\left((\partial_\mu A_\mu + A_\mu \partial_\mu + A_\mu A_\mu)\left(\frac{-1}{\partial^2}\right)\right)_{w_q w_{q+1}} \tag{8.41}$$

in the notation of (1.28).

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